A non-abeliann nonlinear Schrödinger equation and countable superposition of solitons [1]

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Abstract

We study solutions of an operator-valued NLS and apply our results to construct countable superpositions of solitons for the scalar NLS.

2000 MSC: 35Q55, 37K30, 47N20

1 Introduction and main results

In the present note we explain how to construct solutions of the NLS via the study of a corresponding non-abelian system. Non-abelian integrable systems are a very active field of recent research (cf [5, 8, 10, 11] to mention only a few closely related to our work). From our point of view, they provide the appropriate framework to apply functional analytic techniques to the original scalar equations. The operator theoretic part of this note is based on [9], where the whole AKNS is treated uniformly. We point out several sharpenings which become possible for the individual equation at hand.

Concerning applications, we have to restrict to a concise discussion of the countable superpositions of solitons. This topic was initiated by Gesztesy and collaborators with results for several other equations [6, 7]. Their approach requires involved computations along the lines of ISM. Our method leads to much shorter arguments as hard analysis is replaced by advanced functional analysis. For more applications, like the asymptotic description of multipole solutions, the reader is referred to [9].

Replacing u by U_2 and $-\overline{u}$ by U_1 in the scalar NLS $-\mathrm{i}u_t = u_{xx} + 2u|u|^2$ yields a non-abelian NLS system

$$\mathrm{i}U_{1,t} = U_{1,xx} - 2U_1U_2U_1, \quad -\mathrm{i}U_{2,t} = U_{2,xx} - 2U_2U_1U_2 \tag{1.1}$$

We interpret U_1, U_2 as functions depending on the real variables x, t with values in the spaces $\mathcal{L}(E_2, E_1)$, $\mathcal{L}(E_1, E_2)$ of bounded linear operators mapping between Banach spaces E_2 and E_1.

In Section 2, we find soliton-like solutions of (1.1).

Theorem 1. *Let E_1, E_2 be Banach spaces and $A_1 \in \mathcal{L}(E_1)$, $A_2 \in \mathcal{L}(E_2)$. Assume that $L_1 = L_1(x,t) \in \mathcal{L}(E_2, E_1)$, $L_2 = L_2(x,t) \in \mathcal{L}(E_1, E_2)$ are differentiable operator-functions solving the base equations $L_{1,x} = A_1L_1$, $L_{1,t} = -\mathrm{i}A_1^2L_1$, $L_{2,x} = A_2L_2$, $L_{2,t} = \mathrm{i}A_2^2L_2$. Then*

$$U_1 = (I - L_1L_2)^{-1}(A_1L_1 + L_1A_2), \quad U_2 = (I - L_2L_1)^{-1}(A_2L_2 + L_2A_1) \tag{1.2}$$

solve the non-abelian NLS system (1.1) wherever $(I - L_1L_2)$, $(I - L_2L_1)$ are both invertible.

[1]Presented at the 3rd Baltic-Nordic Workshop “Algebra, Geometry, and Mathematical Physics“, Göteborg, Sweden, October 11–13, 2007.

Since the proof of this result is mainly algebraic and does not use specific properties of bounded operators, more general formulations for functions with values in appropriate algebras are possible. We stated the above version for sake of concreteness. The special form of the solution (1.2) was obtained as a consequence of more general considerations about the noncommutative AKNS system in [9]. A reformulation closer to the familiar scalar solution is given in Theorem 3.

In Section 3 we derive solution formulas for the NLS which still depend on arbitrary operator parameters $A_1 \in \mathcal{L}(E_1)$, $A_2 \in \mathcal{L}(E_2)$. Such solution formulas are actually very general (see [2, 4] for the KdV case). As a concrete application, we construct countable superpositions of solitons in Sec. 4.

Theorem 2. *Let $(k_j)_j$ be a bounded sequence with $\mathrm{Re}(k_j) > 0$ for all j, and $(a_j)_j$, $(c_j)_j$ sequences satisfying the growth condition*

$$\left(\frac{a_j}{\sqrt{\mathrm{Re}(k_j)}}\right)_j \in E', \quad \left(\frac{c_j}{\sqrt{\mathrm{Re}(k_j)}}\right)_j \in E \tag{1.3}$$

where E is one of the classical Banach spaces $E = c_0$ or $E = \ell_p$ $(1 \le p < \infty)$ and E' is its topological dual. Then the following spectral determinants of the infinite matrices are well-defined

$$u = 1 - \frac{\det_\lambda \begin{pmatrix} I & -L \\ \overline{L} & I + \overline{L}_0 \end{pmatrix}}{\det_\lambda \begin{pmatrix} I & -L \\ \overline{L} & I \end{pmatrix}}, \quad L = \Big(\frac{\overline{a}_j c_i}{k_i + \overline{k}_j} f_i\Big)_{i,j=1}^{\infty}, \quad L_0 = \Big(a_j c_i f_i\Big)_{i,j=1}^{\infty}$$

where $f_i(x,t) = \exp(k_i x - \mathrm{i}k_i^2 t)$. Moreover, u is a solution of the NLS equation $-\mathrm{i}u_t = u_{xx} + 2u|u|^2$.

If we truncate sequences by requiring $j \le N$, the formula of Theorem 2 describes an N-soliton u_N. Then u is their limit for $N \to \infty$. Notice that neither the existence nor the solution property of u are clear a priori.

2 An operator equation governing the NLS

Let E be a Banach space and $J \in \mathcal{L}(E)$ with $J^2 = -I$. For an operator-valued function $U = U(x,t) \in \mathcal{L}(E)$ we consider the non-commutative partial differential equation

$$-JU_t = U_{xx} - 2U^3 \tag{2.1}$$

and show that it has a traveling wave solution.

Theorem 3. *Let $A \in \mathcal{L}(E)$ with $[A, J] = 0$. Assume that $L = L(x,t) \in \mathcal{L}(E)$ is an operator-valued function anti-commuting with J and solving the base equations $L_x = AL$, $L_t = JA^2L$. Then*

$$U = (I - L^2)^{-1}(AL + LA) \tag{2.2}$$

solves the operator equation (2.1) wherever $I - L^2$ is invertible.

For the proof we introduce in addition the operator-valued function

$$V = (I - L^2)^{-1}(A + LAL) \tag{2.3}$$

Lemma 1. *The derivative of the operator-valued functions $U = U(x,t)$, $V = V(x,t)$ given in (2.2), (2.3) with respect to x is $U_x = VU$, $V_x = U^2$.*

Proof. First we recall the non-abelian differentiation rule for inverse operators. If $T = T(x)$ is differentiable with respect to x and invertible for all $x \in \mathbb{R}$, then T^{-1} is differentiable and $T_x^{-1} = -T^{-1}T_xT^{-1}$. Using the base equations we thus infer

$$\begin{aligned} V_x &= -(I-L^2)^{-1}(-L^2)_x(I-L^2)^{-1}(A+LAL) + (I-L^2)^{-1}(A+LAL)_x \\ &= (I-L^2)^{-1}\Big((AL+LA)L\Big)(I-L^2)^{-1}(A+LAL) + (I-L^2)^{-1}\Big(AL+LA\Big)AL \\ &= U(I-L^2)^{-1}\Big(L(A+LAL)+(I-L^2)AL\Big) \\ &= U(I-L^2)^{-1}\Big(AL+LA\Big) \\ &= U^2 \end{aligned}$$

Analogously, one checks $U_x = VU$. □

In the same way one can calculate the derivative with respect to the time variable. Note that here the fact that $[A,J]=0$ and $\{L,J\}=0$ is crucial. We omit the proof.

Lemma 2. *The derivative of the operator-valued function $U = U(x,t)$ given in (2.2) with respect to t is*

$$U_t = J(I-L^2)^{-1}(A^2 - LA^2L)U$$

Lemma 3. *For the operator-functions U, V in (2.2), (2.3), the following identity holds*

$$V^2 - U^2 = (I-L^2)^{-1}(A^2-LA^2L) \tag{2.4}$$

Proof. We need the following auxiliary identity

$$L(I-L^2)^{-1}L = (I-L^2)^{-1}L^2 = (I-L^2)^{-1}\Big(I-(I-L^2)\Big) = (I-L^2)^{-1} - I$$

which is applied in the third step of the succeeding calculation to replace the terms in the first and in the last large brackets.

$$\begin{aligned} (I-L^2)V^2 &= (A+LAL)(I-L^2)^{-1}(A+LAL) \\ &= A\Big((I-L^2)^{-1}\Big)A + A\Big((I-L^2)^{-1}L\Big)AL + LA\Big(L(I-L^2)^{-1}\Big)A \\ &\quad + LA\Big(L(I-L^2)^{-1}L\Big)AL \\ &= A\Big(I+L(I-L^2)^{-1}L\Big)A + A\Big(L(I-L^2)^{-1}\Big)AL + LA\Big((I-L^2)^{-1}L\Big)A \\ &\quad + LA\Big((I-L^2)^{-1}-I\Big)AL \\ &= (A^2-LA^2L) + (AL+LA)(I-L^2)^{-1}(AL+LA) \\ &= (A^2-LA^2L) + (I-L^2)U^2 \end{aligned}$$

□

Proof of Theorem 3. Using Lemma 1 we get

$$U_{xx} = (VU)_x = V_xU + VU_x = U^3 + V^2U$$

Thus, applying successively Lemma 3 and Lemma 2,

$$U_{xx} - 2U^3 = (V^2-U^2)U = (I-L^2)^{-1}(A^2-LA^2L)U = -JU_t$$

□

Proof of Theorem 1. We obtain Theorem 1 by applying Theorem 3 to $E := E_1 \oplus E_2$,

$$J := \begin{pmatrix} -\mathrm{i}I_{E_1} & 0 \\ 0 & \mathrm{i}I_{E_2} \end{pmatrix}, \quad A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & L_1 \\ L_2 & 0 \end{pmatrix}$$

□

3 Solution formulas for the NLS

Next we explain how Theorem 1 can be used to extract explicit solution formulas for the scalar NLS system

$$\mathrm{i}u_{1,t} = u_{1,xx} - 2u_1^2 u_2, \quad -\mathrm{i}u_{2,t} = u_{2,xx} - 2u_2^2 u_1 \tag{3.1}$$

To formulate our result we recall that a one-dimensional operator $T \in \mathcal{L}(E, F)$ can be written as $a \otimes c$ with appropriate $a \in E'$, $c \in F$, where the map $a \otimes c$ is defined by $a \otimes c(v) = \langle v, a\rangle c$ (and $\langle v, a\rangle$ denotes the evaluation of the functional a on $v \in E$).

Proposition 1. *Let E_1, E_2 be Banach spaces and $A_1 \in \mathcal{L}(E_1)$, $A_2 \in \mathcal{L}(E_2)$. Assume that there are operators $B_1 \in \mathcal{A}(E_2, E_1)$, $B_2 \in \mathcal{A}(E_2, E_1)$, belonging to a quasi-Banach ideal $\mathcal{A}$ admitting a continuous determinant δ, which satisfy the one-dimensionality conditions*

$$A_1B_1 + B_1A_2 = a_2 \otimes c_1, \quad A_2B_2 + B_2A_1 = a_1 \otimes c_2 \tag{3.2}$$

with functionals $a_1 \in E_1'$, $a_2 \in E_2'$, and vectors $c_1 \in E_1$, $c_2 \in E_2$ and $\langle c_1, a_1\rangle = \langle c_2, a_2\rangle = 1$. Then a solution of the NLS system (3.1) is given by

$$u_1 = 1 - \frac{\delta\begin{pmatrix} I - a_1 \otimes \ell_1 & L_1 \\ L_2 & I \end{pmatrix}}{\delta\begin{pmatrix} I & L_1 \\ L_2 & I \end{pmatrix}}, \quad u_2 = 1 - \frac{\delta\begin{pmatrix} I & L_1 \\ L_2 & I - a_2 \otimes \ell_2 \end{pmatrix}}{\delta\begin{pmatrix} I & L_1 \\ L_2 & I \end{pmatrix}} \tag{3.3}$$

with the operator-functions $L_j(x,t) = \widehat{L}_j(x,t)B_j$, the vector-functions $\ell_j(x,t) = \widehat{L}_j(x,t)c_j$, where $\widehat{L}_1(x,t) = \exp(A_1x - \mathrm{i}A_1^2t)$, $\widehat{L}_2(x,t) = \exp(A_2x + \mathrm{i}A_2^2t)$ provided the denominator does not vanish.

Remark 1. We want to stress that the one-dimensionality condition (3.2) can always be met provided $0 \notin \mathrm{spec}(A_1) + \mathrm{spec}(A_2)$ (Minkowski sum), see also [3]. The normalization $\langle c_j, a_j\rangle = 1$ is only chosen for convenience. It suffices to assume $\langle c_j, a_j\rangle \neq 0$ for $j = 1, 2$.

Proof. The main argument of the proof is contained in the Step 2 where a solution of the scalar system is constructed from a solution of the non-abelian system by cross-evaluation.

Step 1: Applying Theorem 1, it can be immediately checked that the operator-functions

$$U_1 = (I - L_1L_2)^{-1}(A_1L_1 + L_1A_2), \quad U_2 = (I - L_2L_1)^{-1}(A_2L_2 + L_2A_1)$$

solve the non-abelian NLS system (1.1).

Step 2: As a consequence of the one-dimensionality condition (3.2),

$$U_1 = (I - L_1L_2)^{-1}\widehat{L}_1(A_1B_1 + B_1A_2) = (I - L_1L_2)^{-1}\widehat{L}_1(a_2 \otimes c_1) = a_2 \otimes f_1$$

where $f_1 = (I - L_1L_2)^{-1}\widehat{L}_1c_1$. Similarly, $U_2 = a_1 \otimes f_2$ with $f_2 = (I - L_2L_1)^{-1}\widehat{L}_2c_2$.

We now show that

$$u_1 = \langle f_1, a_1\rangle, \quad u_2 = \langle f_2, a_2\rangle \tag{3.4}$$

solve (3.1). Indeed, evaluating the first equation of the operator system (1.1) on the vector c_2, we obtain the vector-equation

$$f_{1,xx} - 2\langle f_1, a_1\rangle\langle f_2, a_2\rangle f_1 = \mathrm{i}f_{1,t}$$

Applying the functional a_1, we get the first equation of the system (3.1). Similarly for the second equation of (3.1).

Step 3: It remains to verify the solution formula in terms of the determinant available on the underlying quasi-Banach ideal. To this end we first note

$$\begin{pmatrix} I & L_1 \\ L_2 & I \end{pmatrix}^{-1} = \begin{pmatrix} (I-L_1L_2)^{-1} & 0 \\ 0 & (I-L_2L_1)^{-1} \end{pmatrix} \begin{pmatrix} I & -L_1 \\ -L_2 & I \end{pmatrix}$$

Using the multiplicity property of a determinant and the fact that on the finite-dimensional operators the generalized determinant coincides with the standard determinant[2],

$$\begin{aligned}
\delta\begin{pmatrix} I-a_1\otimes\ell_1 & L_1 \\ L_2 & I \end{pmatrix} \Big/ \delta\begin{pmatrix} I & L_1 \\ L_2 & I \end{pmatrix} &= \delta\left(\begin{pmatrix} I & L_1 \\ L_2 & I \end{pmatrix}^{-1}\begin{pmatrix} I-a_1\otimes\ell_1 & L_1 \\ L_2 & I \end{pmatrix}\right) \\
&= \delta\begin{pmatrix} I-(I-L_1L_2)^{-1}a_1\otimes\ell_1 & 0 \\ (I-L_2L_1)^{-1}L_2a_1\otimes\ell_1 & I \end{pmatrix} \\
&= \delta\Big(I-\big(a_1,0\big)\otimes\big((I-L_1L_2)^{-1}\ell_1, -(I-L_2L_1)^{-1}L_2\ell_1\big)\Big) \\
&= 1-\langle (I-L_1L_2)^{-1}\ell_1, a_1\rangle \quad = \quad 1-u_1
\end{aligned}$$

and similarly for the other identity in (3.3). □

4 Countable superposition of solitons

As an application we study solutions of the NLS arising from our solution formula by plugging in diagonal operators on sequence spaces. The resulting solution class describes the superposition of countably many solitons.

Choose $E_1 = E_2 =: E$ to be one of the classical sequence spaces c_0, ℓ_p $(1 \le p < \infty)$ and, for a given bounded sequence $(k_j)_j$ with $\mathrm{Re}(k_j) > 0$ for all j, we define A_1, A_2 to be the diagonal operators generated by $(k_j)_j$, $(\overline{k}_j)_j$, respectively.

Let $a_1 =: a$, $c_1 =: c$ be sequences satisfying the growth condition (1.3) and set $a_2 = -\overline{a}_1$, $c_2 = \overline{c}_1$. Then the one-dimensionality condition (3.2) can be solved explicitly by

$$B_1 = \Big(\frac{-\overline{a}_j c_i}{k_i + \overline{k}_j}\Big)_{i,j=1}^{\infty} \qquad \in \mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1(E)$$

and $B_2 = -\overline{B}_1$, where $\mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1$ denotes the quasi-Banach ideal of operators factorizing through first an L_1-space, then a Hilbert space, and finally an L_∞-space.

Indeed, $B_1 = -D_2T'TD_1$ where $D_1 : E \to \ell_1$, $D_2 : \ell_\infty \to E$ are the diagonal operators generated by the sequences $(\overline{a}_j/\sqrt{\mathrm{Re}(k_j)})_j$, $(c_j/\sqrt{\mathrm{Re}(k_j)})_j$, and $T : \ell_1 \to L_2[0,\infty)$ defined on the standard basis by $Te_j = \overline{f}_j$ with $f_j(s) = \sqrt{\mathrm{Re}(k_j)}\exp(-k_j s)$.

Since $\mathcal{L}_\infty \circ \mathcal{H} \circ \mathcal{L}_1$ admits a continuous (even spectral) determinant $\det_\lambda$ [1], the solution formula of Proposition 1 can be applied. Moreover one can check that the particular choices above guarantee $u := u_2 = -\overline{u}_1$. This yields Theorem 2.

Remark 2. The results of this section can be easily generalized to construct also countable superpositions of multipole solutions of the NLS. Moreover it can be shown that all these solutions are globally regular. For details see [9].

Acknowledgement

I would like to thank the referee for careful reading and valuable remarks leading to a more suggestive presentation of the main results.

[2]in particular $\delta(I + a\otimes c) = \det(I + a\otimes c) = 1 + \langle c, a\rangle$ for one-dimensional endomorphisms $a\otimes c$

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Received December 17, 2007
Revised March 27, 2008