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Classification of Canonical Bases for (*n*–1)–dimensional Subspaces of *n*–Dimensional Vector Space

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Abstract

Canonical bases for (*n*-1)-dimensional subspaces of n-dimensional vector space are introduced and classified in the article. This result is very prospective to utilize canonical bases at all applications. For example, maximal subalgebras of Lie algebras can be found using them.

Keywords: Vector space; Subspaces; Canonical bases

Introduction

The canonical bases for (n-1)-dimensional subspaces of n–dimensional vector space are introduced in the article, and all nonequivalent of them are classified (Theorem 2). This result generalizes a particular result for 5-dimensional subspaces of 6-dimensional vector space obtained in the previous article of the same author. To analyze the general case, reduced row echelon forms of matrices are utilized; about reduced row echelon forms [1]. In addition to the principal result, all nonequivalent reduced row echelon forms for $(n-1)\times n$ matrices of the rank $(n-1)$ are found and listed (Theorem 1).

Let V be an $n-$ dimensional vector space with its standard basis $\vec{e}_1,\vec{e}_2,\dots,\vec{e}_n$. Let $\vec{a}_1,\vec{a}_2,\dots,\vec{a}_{n-1}$ be n – 1 linearly independent vectors in the space V where

$$\vec{a}_1=a_{11}\vec{e}_1+a_{12}\vec{e}_2+\dots+a_{1n}\vec{e}_n,\ \vec{a}_2=a_{21}\vec{e}_1+a_{22}\vec{e}_2+\dots+a_{2n}\vec{e}_n,\dots,\vec{a}_{n-1}=a_{n-1,1}\vec{e}_1+\dots+a_{n-1,n}\vec{e}_n. \quad \text{(I)}$$

The vectors (I) describe the possible bases for any V -dimensional subspace S of V.

Definition 1

Two bases are called **equivalent** if they generate the same subspace of V, and they are called **nonequivalent** if they generate two different subspaces of V.

We will associate the following $(n-1)\times n$ matrix M with a basis (I)

$$M=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & .. & .. \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n} \end{bmatrix} \quad \text{(II)}$$

Definition 2

Two matrices are called **row equivalent** (or just **equivalent)** if they have the same reduced row echelon form, and they are called **nonequivalent** if they have different reduced row echelon forms.

Definition 3

The basis (I) is called **canonical** if its vectors $\vec{a}_1,\vec{a}_2,\dots,\vec{a}_{n-1}$ are the corresponding rows in some reduced row echelon form of the matrix M.

Thus, there is one-to-one correspondence between nonequivalent canonical bases for $(n-1)$-dimensional subspaces of n–dimensional vector space and reduced row echelon forms for $(n-1)\times n$ matrix M.

Remark

The standard linear operations with rows (vectors) will be utilized:

(a) interchange any two rows, (b) multiply any row by a nonzero constant, (c) add a multiple of some row to another row.

Consider some examples with nonequivalent canonical bases for subspaces of small dimensional vector spaces.

Ex 1

Let V be 2-dimensional vector space with its standard basis $\vec{e}_1$, $\vec{e}_2$. Each 1-dimensional subspace S of V can be described as $S=Span\{\vec{a}_1\}$ where $\vec{a}_1=a_{11}\vec{e}_1+a_{12}\vec{e}_2$. At least one component among a_{11}, a_{12} of the vector $\vec{a}_1$ is not zero. If $a_{12}\neq 0$ then perform the operation $\vec{a}_1/a_{12}$, and we obtain the first canonical basis $\{\vec{e}_1+a_{12}\vec{e}_2\}$. If $a_{12}=0$ then perform the similar operation $\vec{a}_1/a_{12}$, and we obtain the second canonical basis $\{\vec{e}_2\}$. These two canonical bases are nonequivalent [2].

Ex 2

Let V be 3-dimensional vector space with its standard basis $\vec{e}_1,\vec{e}_2,\vec{e}_3$. Consider any 2-dimensional subspace S of V that can be described as $S=Span\{\vec{a}_1,\vec{a}_2\}$ where

$$\vec{a}_1=a_{11}\vec{e}_1+a_{12}\vec{e}_2+a_{13}\vec{e}_3,\ \vec{a}_2=a_{21}\vec{e}_1+a_{22}\vec{e}_2+a_{23}\vec{e}_3.$$

This basis is equivalent to one and only one canonical basis from the next list

1. $\vec{a}_1=\vec{e}_1+a_{13}\vec{e}_3, \vec{a}_2=\vec{e}_2+a_{23}\vec{e}_3$. **2.** $\vec{a}_1=\vec{e}_1+a_{12}\vec{e}_2, \vec{a}_2=\vec{e}_3$. **3.** $\vec{a}_1=\vec{e}_2, \vec{a}_2=\vec{e}_3$.

Details of evaluation are omitted because it is easy. These last canonical bases generate the following matrices associated with them

$$(1)\begin{bmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \end{bmatrix},\ (2)\begin{bmatrix} 1 & a_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ (3)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Ex 3

Let V be 6-dimensional vector space with its standard basis, ,

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$\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5, \vec{e}_6$. Consider any 5-dimensional subspace S of V that can be described as $S = Span\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\}$ where

$$\vec{a}_1 = a_{11}\vec{e}_1 + a_{12}\vec{e}_2 + a_{13}\vec{e}_3 + a_{14}\vec{e}_4 + a_{15}\vec{e}_5 + a_{16}\vec{e}_6,\ \vec{a}_2 = a_{21}\vec{e}_1 + a_{22}\vec{e}_2 + a_{23}\vec{e}_3 + a_{24}\vec{e}_4 + a_{25}\vec{e}_5 + a_{26}\vec{e}_6,$$

$$\vec{a}_3 = a_{31}\vec{e}_1 + a_{32}\vec{e}_2 + a_{33}\vec{e}_3 + a_{34}\vec{e}_4 + a_{35}\vec{e}_5 + a_{36}\vec{e}_6,\ \vec{a}_4 = a_{41}\vec{e}_1 + a_{42}\vec{e}_2 + a_{43}\vec{e}_3 + a_{44}\vec{e}_4 + a_{45}\vec{e}_5 + a_{46}\vec{e}_6$$

$$\vec{a}_5 = a_{51}\vec{e}_1 + a_{52}\vec{e}_2 + a_{53}\vec{e}_3 + a_{54}\vec{e}_4 + a_{55}\vec{e}_5 + a_{56}\vec{e}_6 .$$

These bases can be transformed into one and only one canonical basis from the next list

$$\vec{a}_1 = \vec{e}_1 + a_{16}\vec{e}_6,\ \vec{a}_2 = \vec{e}_2 + a_{26}\vec{e}_6,\ \vec{a}_3 = \vec{e}_3 + a_{36}\vec{e}_6,\ \vec{a}_4 = \vec{e}_4 + a_{46}\vec{e}_6, \vec{a}_5 = \vec{e}_5 + a_{56}\vec{e}_6;\quad (a_1)$$

$$\vec{a}_1 = \vec{e}_1 + a_{15}\vec{e}_5,\ \vec{a}_2 = \vec{e}_2 + a_{25}\vec{e}_5, \vec{a}_3 = \vec{e}_3 + a_{35}\vec{e}_5, \vec{a}_4 = \vec{e}_4 + a_{45}\vec{e}_5,\ \vec{a}_5 = \vec{e}_6;\quad (a_2)$$

$$\vec{a}_1 = \vec{e}_1 + a_{14}\vec{e}_4, \vec{a}_2 = \vec{e}_2 + a_{24}\vec{e}_4, \vec{a}_3 = \vec{e}_3 + a_{34}\vec{e}_4,\ \vec{a}_4 = \vec{e}_5, \vec{a}_5 = \vec{e}_6;\quad (a_3)$$

$$\vec{a}_1 = \vec{e}_1 + a_{13}\vec{e}_3,\ \vec{a}_2 = \vec{e}_2 + a_{23}\vec{e}_3,\ \vec{a}_3 = \vec{e}_4,\ \vec{a}_4 = \vec{e}_5, \vec{a}_5 = \vec{e}_6;\quad (a_4)$$

$$\vec{a}_1 = \vec{e}_1 + a_{12}\vec{e}_2,\ \vec{a}_2 = \vec{e}_3,\ \vec{a}_3 = \vec{e}_4,\ \vec{a}_4 = \vec{e}_5,\ \vec{a}_5 = \vec{e}_6;\quad (a_5)$$

$$\vec{a}_1 = \vec{e}_2, \vec{a}_2 = \vec{e}_3, \vec{a}_3 = \vec{e}_4, \vec{a}_4 = \vec{e}_5,\ \vec{a}_5 = \vec{e}_6\ .\quad (a_6)$$

All these canonical bases are nonequivalent. This result is obtained by the direct evaluation that generalizes Gauss-Jordan elimination method. The necessary details can be found in the different article of the same author [3].

The following matrices are associated with the last canonical bases $(a_1) - (a_6)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 1 & 0 & 0 & 0 & a_{26} \\ 0 & 0 & 1 & 0 & 0 & a_{36} \\ 0 & 0 & 0 & 1 & 0 & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & a_{15} & 0 \\ 0 & 1 & 0 & 0 & a_{25} & 0 \\ 0 & 0 & 1 & 0 & a_{35} & 0 \\ 0 & 0 & 0 & 1 & a_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & a_{14} & 0 & 0 \\ 0 & 1 & 0 & a_{24} & 0 & 0 \\ 0 & 0 & 1 & a_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & a_{13} & 0 & 0 & 0 \\ 0 & 1 & a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrices obtained in **Examples 2** and **3** are particular cases of the following matrices described by the next statement.

Theorem 1

All nonequivalent reduced row echelon forms of $(n-1)\times n$ matrices (n 3) of the rank $(n-1)$ are

$$\textbf{(1)} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & a_{1n} \\ 0 & 1 & 0 & \ldots & 0 & a_{2n} \\ 0 & 0 & 1 & \ldots & 0 & a_{3n} \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & a_{n-1,n} \end{bmatrix}, \quad \textbf{(2)} \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & a_{1,n-1} & 0 \\ 0 & 1 & 0 & \ldots & 0 & a_{2,n-1} & 0 \\ 0 & 0 & 1 & \ldots & 0 & a_{3,n-1} & 0 \\ . & . & . & \ldots & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & a_{n-2,n-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix},$$

$$\textbf{(3)} \begin{bmatrix} 1 & 0 & \ldots & 0 & a_{1,n-2} & 0 & 0 \\ 0 & 1 & \ldots & 0 & a_{2,n-2} & 0 & 0 \\ . & . & \ldots & . & . & . & . \\ 0 & 0 & \ldots & 1 & a_{n-3,n-2} & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \textbf{(4)} \begin{bmatrix} 1 & 0 & \ldots & 0 & a_{1,n-3} & 0 & 0 & 0 \\ 0 & 1 & \ldots & 0 & a_{2,n-3} & 0 & 0 & 0 \\ . & . & \ldots & . & . & . & . & . \\ 0 & 0 & \ldots & 1 & a_{n-4,n-3} & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

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$$\textbf{(n-1)} \begin{bmatrix} 1 & a_{12} & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ . & . & . & . & \ldots & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{bmatrix}, \quad \textbf{(n)} \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & . & \ldots & . & . \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

Proof

We use the mathematical induction method. The statement is true for n=3 and n=6 according **Examples 2** and **3.**

Suppose that the statement is true for some dimension n, and prove it for the next dimension n + 3. For it, consider the following matrix of the size $n \times (n+1)$

$$M' = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \ldots & a_{2n} & a_{2,n+1} \\ . & . & \ldots & . & . \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n} & a_{n-1,n+1} \\ a_{n1} & a_{n2} & \ldots & a_{n,n} & a_{n,n+1} \end{bmatrix}.$$

Go to the $(n-1) \times n$ submatrix located in the left upper corner of M'. According the assumption, this submatrix can be transformed into one and only one reduced row echelon form described in the cases (1), (2), (3), (4), ….(n–1), (n)**.** This means that we can replace the mentioned submatrix by one of the given reduced forms, and analyze the new matrix. We will analyze and show all details in steps 1, 2, 3, and steps (n-1), (n). All other steps are very similar to the steps 1, 2, 3, therefore they are omitted.

Step 1

Substitute the left upper $(n-1) \times n$ submatrix in M by the matrix **(1).** We have the following matrix

$$M' = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & a_{1n} & a_{1,n+1} \\ 0 & 1 & 0 & \ldots & 0 & a_{2n} & a_{2,n+1} \\ 0 & 0 & 1 & \ldots & 0 & a_{3n} & a_{3,n+1} \\ . & . & . & \ldots & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & a_{n-1,n} & a_{n-1,n+1} \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{n,n-1} & a_{n,n} & a_{n,n+1} \end{bmatrix}.$$

Perform the following operations $\vec{a}_n - a_{n1}\vec{a}_1,\ \vec{a}_n - a_{n2}\vec{a}_2, \ldots, \vec{a}_n - a_{n,n-1}\vec{a}_{n-1}$. We obtain

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & a_{1n} & a_{1,n+1} \\ 0 & 1 & 0 & \ldots & 0 & a_{2n} & a_{2,n+1} \\ 0 & 0 & 1 & \ldots & 0 & a_{3n} & a_{3,n+1} \\ . & . & . & \ldots & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & a_{n-1,n} & a_{n-1,n+1} \\ 0 & 0 & 0 & \ldots & 0 & a_{n,n} & a_{n,n+1} \end{bmatrix}.$$

Consider the elements $a_{n,n}$, $a_{n,\ n+1}$ in the last matrix. At least one of them is not zero because the rank of the matrix M' is equal n. If $a_{n,n} \neq 0$ then perform the operation $\vec{a}_n / a_{n,n}$ first, and the operations $\vec{a}_1 - a_{1n}\vec{a}_n, \ldots, \vec{a}_{n-1} - a_{n-1,n}\vec{a}_n$ after the first one. We obtained the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & a_{1,n+1} \\ 0 & 1 & 0 & \ldots & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 1 & \ldots & 0 & 0 & a_{3,n+1} \\ . & . & . & \ldots & . & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 & a_{n-1,n+1} \\ 0 & 0 & 0 & \ldots & 0 & 1 & a_{n,n+1} \end{bmatrix}.$$

This is the matrix of the type (1) as it's needed. If $a_{n,\,n} = 0$ then $a_{n,\,n+1} \neq 0$. Perform the operation $\vec{a}_n / a_{n,n+1}$ first, and the operations $\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$. We obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & a_{1n} & 0 \\ 0 & 1 & 0 & \dots & 0 & a_{2n} & 0 \\ 0 & 0 & 1 & \dots & 0 & a_{3n} & 0 \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & 1 & a_{n-1,n} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

This is the matrix of the type (2) as it's needed. Step 1 is done.

Step 2

Substitute the left upper $(n-1) \times n$ submatrix in M by the matrix **(2).** We have

$$M' = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & a_{1,n-1} & 0 & a_{1,n+1} \\ 0 & 1 & 0 & 0 & \dots & a_{2,n-1} & 0 & a_{2,n+1} \\ 0 & 0 & 1 & 0 & \dots & a_{3,n-1} & 0 & a_{3,n+1} \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & a_{n-2,n-1} & 0 & a_{n-2,n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n+1} \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \dots & a_{n,n-1} & a_{nn} & a_{n,n+1} \end{bmatrix}.$$

Perform the operations, $\vec{a}_n - a_{n1}\vec{a}_1, \vec{a}_n - a_{n2}\vec{a}_2, \dots, \vec{a}_n - a_{n,n-2}\vec{a}_{n-2}, \vec{a}_n - a_{n,n}\vec{a}_{n-1}$. We obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & a_{1,n-1} & 0 & a_{1,n+1} \\ 0 & 1 & 0 & 0 & \dots & a_{2,n-1} & 0 & a_{2,n+1} \\ 0 & 0 & 1 & 0 & \dots & a_{3,n-1} & 0 & a_{3,n+1} \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & a_{n-2,n-1} & 0 & a_{n-2,n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n+1} \\ 0 & 0 & 0 & 0 & \dots & a_{n,n-1} & 0 & a_{n,n+1} \end{bmatrix}.$$

At least one element among $a_{n,\,n1}$, $a_{n,\,n+1}$ is not zero. If $a_{n,\,n1} \neq 0$ then perform the operation $\vec{a}_n / a_{n,n-1}$ first, and the operations $\vec{a}_1 - a_{1,n-1}\vec{a}_n, \vec{a}_2 - a_{2,n-1}\vec{a}_n, \dots, \vec{a}_{n-2} - a_{n-2,n-1}\vec{a}_n$ after the first operation. We obtain the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & a_{1,n+1} \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & a_{3,n+1} \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_{n-2,n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n+1} \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & a_{n,n+1} \end{bmatrix}.$$

The last matrix is row equivalent to the matrix (1) as it's needed. If $a_{n,\,n-1}=0$ then $a_{n,\,n+1} \neq 0$. Perform the operation $\vec{a}_n / a_{n,n+1}$ first, and the operations $\vec{a}_1 - a_{1,n+1}\vec{a}_n, \dots, \vec{a}_{n-1} - a_{n-1,n+1}\vec{a}_n$ after the first one. We obtain the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & a_{1,n-1} & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & a_{2,n-1} & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & a_{3,n-1} & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & a_{n-2,n-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

The last matrix is a matrix of the type (3) as it's needed.

Step 3

Substitute the left upper $(n-1) \times n$ submatrix in M by the matrix **(3).** We have

$$M' = \begin{bmatrix} 1 & 0 & \dots & 0 & a_{1,n-2} & 0 & 0 & a_{1,n+1} \\ 0 & 1 & \dots & 0 & a_{2,n-2} & 0 & 0 & a_{2,n+1} \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & \dots & 1 & a_{n-3,n-2} & 0 & 0 & a_{n-3,n+1} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & a_{n-2,n+1} \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & a_{n-1,n+1} \\ a_{n,1} & a_{n,2} & a_{n.3} & a_{n,n-3} & a_{n,n-2} & a_{n,n-1} & a_{n,n} & a_{n,n+1} \end{bmatrix}.$$

Perform the operations $\vec{a}_n - a_{n1}\vec{a}_1, \vec{a}_n - a_{n2}\vec{a}_2, \dots, \vec{a}_n - a_{n,n-2}\vec{a}_{n-2}, \vec{a}_n - a_{n,n}\vec{a}_{n-1}$. We obtain

$$\begin{bmatrix} 1 & 0 & \dots & 0 & a_{1,n-2} & 0 & 0 & a_{1,n+1} \\ 0 & 1 & \dots & 0 & a_{2,n-2} & 0 & 0 & a_{2,n+1} \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & \dots & 1 & a_{n-3,n-2} & 0 & 0 & a_{n-3,n+1} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & a_{n-2,n+1} \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & a_{n-1,n+1} \\ 0 & 0 & 0 & 0 & a_{n,n-2} & 0 & 0 & a_{n,n+1} \end{bmatrix}.$$

At least one element among $a_{n,\,n2}$, $a_{n,\,n+1}$ is not zero in this matrix. If $a_{n,\,n2}$ then perform the operation $\vec{a}_n / a_{n,n-2}$ first, and the operations $\vec{a}_1 - a_{1,n-2}\vec{a}_n, \vec{a}_2 - a_{2,n-2}\vec{a}_n, \dots, \vec{a}_{n-3} - a_{n-3,n-2}\vec{a}_n$ after the first operation. We obtain the following matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & 0 & a_{1,n+1} \\ 0 & 1 & \dots & 0 & 0 & 0 & 0 & a_{2,n+1} \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & a_{n-3,n+1} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & a_{n-2,n+1} \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & a_{n-1,n+1} \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & a_{n,n+1} \end{bmatrix}.$$

The last matrix is row equivalent to the matrix (1) as it's needed. If $a_{n,\,n-2}= 0$ then $a_{n,\,n+1} \neq 0$. Perform the operation $\vec{a}_n / a_{n,n+1}$ first, and the operations $\vec{a}_1 - a_{1,n+1}\vec{a}_n, \dots, \vec{a}_{n-1} - a_{n-1,n+1}\vec{a}_n$ after the first one. We obtain the following matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 & a_{1,n-2} & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & a_{2,n-2} & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & \dots & 1 & a_{n-3,n-2} & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have the matrix that is equivalent to the matrix of the type (4) as it's needed.

Step $(n-1)$. Substitute the left upper $(n-1) \times n$ submatrix in M' by the matrix $(n-1)$. We have

$$M'=\begin{bmatrix} 1 & a_{12} & 0 & 0 & \dots & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & a_{3,n+1} \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & a_{n-2,n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n+1} \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \dots & a_{n,n-1} & a_{n,n} & a_{n,n+1} \end{bmatrix}.$$

Perform the operations $\overrightarrow{a_n}-a_{n,1}\overrightarrow{a_1},\overrightarrow{a_n}-a_{n,3}\overrightarrow{a_2},\dots,\overrightarrow{a_n}-a_{n,n-1}\overrightarrow{a_{n-2}},\overrightarrow{a_n}-a_{n,n}\overrightarrow{a_{n-1}}$. We obtain

$$\begin{bmatrix} 1 & a_{12} & 0 & 0 & \dots & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & a_{3,n+1} \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & a_{n-2,n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n+1} \\ 0 & a_{n,2} & 0 & 0 & \dots & 0 & 0 & a_{n,n+1} \end{bmatrix}.$$

At least one element among $a_{n,2}$, $a_{n,n+1}$ is not zero in this matrix. If $a_{n,2}\neq 0$ then perform the operation $\overrightarrow{a_n}/a_{n,2}$ first, and the operation $\overrightarrow{a_1}-a_{12}\overrightarrow{a_n}$ after the first operation. We obtain the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & a_{3,n+1} \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & a_{n-2,n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n+1} \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & a_{n,n+1} \end{bmatrix}.$$

If interchange rows in the last matrix, we obtain the matrix **(1)** as it's needed. If $a_{n,2}=0$ then $a_{n,n+1}\neq 0$. Perform the operation $\overrightarrow{a_n}/a_{n,n+1}$ first, and the operations $\overrightarrow{a_1}-a_{1,n+1}\overrightarrow{a_n},\dots,\overrightarrow{a_{n-1}}-a_{n-1,n+1}\overrightarrow{a_n}$ after the first one. We obtain the following matrix

$$\begin{bmatrix} 1 & a_{12} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

It is the matrix of the type **(n-1)** as we need.

Step n

Substitute the left upper $(n-1)\times n$ submatrix in M by the matrix **(n).** We have

$$M'=\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & a_{3,n+1} \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & a_{n-2,n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n+1} \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \dots & a_{n,n-1} & a_{n,n} & a_{n,n+1} \end{bmatrix}.$$

Perform the operations $\overrightarrow{a_n}-a_{n,2}\overrightarrow{a_1},\overrightarrow{a_n}-a_{n,3}\overrightarrow{a_2},\dots,\overrightarrow{a_n}-a_{n,n-1}\overrightarrow{a_{n-2}},\overrightarrow{a_n}-a_{n,n}\overrightarrow{a_{n-1}}$ We obtain

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & a_{1,n+1} \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & a_{2,n+1} \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & a_{3,n+1} \\ . & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & a_{n-2,n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_{n-1,n+1} \\ a_{n,1} & 0 & 0 & 0 & \dots & 0 & 0 & a_{n,n+1} \end{bmatrix}.$$

At least one element among $a_{n,1}$, $a_{n,n+1}$ is not zero in this matrix. If $a_{n,1}\neq 0$ then perform the operation $\overrightarrow{a_n}/a_{n,1}$ and interchange the rows. We obtain the matrix of the type **(1)** as it's needed. If $a_{n,1}=0$ then $a_{n,n+1}$0. Perform the operation $\overrightarrow{a_n}/a_{n,n+1}$ first, and the operations, $\overrightarrow{a_1}-a_{1,n+1}\overrightarrow{a_n},\dots,\overrightarrow{a_{n-1}}-a_{n-1,n+1}\overrightarrow{a_n}$ after the first one. We obviously obtain the matrix of the type **(n)** as it's needed. The proof is done.

Theorem 1 can be transformed into the next statement that describes canonical bases for $(n-1)$ –dimensional subspaces of n–dimensional vector space.

Theorem 2

All nonequivalent canonical bases for $(n-1)$ –dimensional subspaces of n–dimensional vector space (if ($n\geq 2$)) are

(1) $\overrightarrow{a_1}=\overrightarrow{e_1}+a_{1n}\overrightarrow{e_n},\ \overrightarrow{a_2}=\overrightarrow{e_2}+a_{2n}\overrightarrow{e_n},\overrightarrow{a_3}=\overrightarrow{e_3}+a_{3n}\overrightarrow{e_n},\dots,\overrightarrow{a_{n-1}}=\overrightarrow{e_{n-1}}+a_{n-1,n}\overrightarrow{e_n};$

(2) $\overrightarrow{a_1}=\overrightarrow{e_1}+a_{1,n-1}\overrightarrow{e_{n-1}},\overrightarrow{a_2}=\overrightarrow{e_2}+a_{2,n-1}\overrightarrow{e_{n-1}},\dots,\overrightarrow{a_{n-2}}=\overrightarrow{e_{n-2}}+a_{n-2,n-1}\overrightarrow{e_{n-1}},\overrightarrow{a_{n-1}}=\overrightarrow{e_n};$

(3) $\overrightarrow{a_1}=\overrightarrow{e_1}+a_{1,n-2}\overrightarrow{e_{n-2}},\ \overrightarrow{a_2}=\overrightarrow{e_2}+a_{2,n-2}\overrightarrow{e_{n-2}},\dots,\ \overrightarrow{a_{n-3}}=\overrightarrow{e_{n-3}}+a_{n-3,n-2}\overrightarrow{e_{n-2}},\overrightarrow{a_{n-2}}=\overrightarrow{e_{n-1}},\overrightarrow{a_{n-1}}=\overrightarrow{e_n};$

. . . .

(n-1) $\overrightarrow{a_1}=\overrightarrow{e_1}+a_{12}\overrightarrow{e_2},\ \overrightarrow{a_2}=\overrightarrow{e_3},\overrightarrow{a_3}=\overrightarrow{e_4},\dots,\overrightarrow{a_{n-3}}=\overrightarrow{e_{n-2}},\overrightarrow{a_{n-2}}=\overrightarrow{e_{n-1}},\overrightarrow{a_{n-1}}=\overrightarrow{e_n};$

(n) $\overrightarrow{a_1}=\overrightarrow{e_2},\overrightarrow{a_2}=\overrightarrow{e_3},\overrightarrow{a_3}=\overrightarrow{e_4},\dots,\overrightarrow{a_{n-3}}=\overrightarrow{e_{n-2}},\overrightarrow{a_{n-2}}=\overrightarrow{e_{n-1}},\overrightarrow{a_{n-1}}=\overrightarrow{e_n}.$

The statement is true for $n\geq 3$ because of Theorem 1. The additional case $n = 2$ is included in Theorem 2 because of **Example 1.**

Final Remark

The canonical bases introduced in the article are a powerful instrument that can be utilized in all applications of Linear Algebra. For example, all maximal subalgebras of any Lie algebra can be found using canonical bases listed in Theorem 2.

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