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Existence and Uniqueness of Asymptotically *w*-Periodic Solution for Fractional Semilinear Problem

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Abstract

In this paper, we intend to show the fractional differential problem $D_t^\alpha u(t) = A(t)u(t) + f(t,u(t))$, with condition 0<α<1, considered in a Banach space *X*, where *A* is a generator of evolution system U(*t*,*s*) and *f* is *w*-periodic limit function, has a unique asymptotically *w*-periodic solution.

Keywords: Asymptotically *w*-periodic solution; Semilinear integro-differential equation; Evolution system

Introduction

After the s-asymptotically *w*-periodic functions in Banach space has been studied for the first time by Henriquez et al. [1], the existence of such solutions for fractional equation was the focus of the attention of the various authors [2,3].

For instance, Jia, et al. [4] studied the existence and uniqueness of periodic solutions, s-asymptotically periodic solutions and other types of bounded solutions for fractional evolution equation:

$$D_+^\alpha u(t) + Au(t) = f(t,u(t)), \qquad t \in R,$$

where D_+^α is the Weyl-Liouville fractional derivative of order α∈(0,1) and –A:D(*A*)⊂*X*→*X* is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$.

By introducing the *w*-periodic limit functions by Xi [5], they investigated the existence and uniqueness of asymptotically *w*-periodic solutions for the following abstract Cauchy problems:

$$x'(t) = Ax(t) + f(t,x(t)), \qquad t \in R^+,$$

$$x(0)=x_0 \in X$$

where *A* infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ and *f* is *w*-periodic limit in t∈R⁺ uniformly for *x* in bounded subsets of *X*.

In this paper, we consider the fractional semilinear problem:

$$\begin{cases} x^\alpha(t) = A(t)x(t) + f(t,x(t)), \\ x(0) = x_0 \in X, \end{cases} \tag{1}$$

where t∈R⁺, where D_+^α is the Riemann-Liouville fractional derivative of order α∈(0,1), and A(*t*):D_t⊂*X*→*X* is a generator an evolution family $(U(t,s))_{t\geq s\geq 0}$ on *X* and *f* is w-periodic limit function and study the existence and uniqueness of asymptotically *w*-periodic solution.

Jawahdou [6] studied the existence of mild solutions of fractional semilinear integro-differential equation:

$$\begin{cases} x^\alpha(t) = A(t)x(t) + f(t,x(t),\int_0^t u(t,s,x(s))ds), \\ x(0) = x_0 \in X, \end{cases} \tag{2}$$

where t>0,0<α<1and *A*(*t*):D_t⊂*X*→*X* generates an evolution system *U*(*t*,*s*).

In section 4, we consider the problem (2) where *f* and *u* are *w*-periodic limit functions and study the existence and uniqueness of asymptotically *w*-periodic solution.

Preliminaries

In this section, we describe a few definitions and propositions that are needed to achieve our result. Let (*X*,||.||) is a Banach space and $C_b(R^+,X)$ the space consisting of bounded and continuous functions from R^+ into *X*, endowed with the uniform convergence norm $||.||_\infty$. Let:

$$C_0(R^+,X) = \{f \in C_b(R^+,X) : \lim_{t\to 0} \| f(t) \| = 0\},$$

$$P_w(R^+,X) = \{f \in C_b(R^+,X) : f\, is\, w - periodic\}.$$

Definition 2.1

A function $f \in C_b(R^+,X)$ is said to be asymptotically *w*-periodic if it can be expressed az *f*=*g*+*h*, where $g \in P_w(R^+,X)$ and $h \in C_0(R^+,X)$. The subspace of $C_b(R^+,X)$ consisting of the asymptotically *w*-periodic functions will be denoted by $AP_w(R^+,X)$.

Definition 2.2

Let $f \in C_b(R^+,X)$ and *w>0*, we call *f* *w*-periodic limit if $g(t) = \lim_{n\to\infty} f(t+nw)$ is well defined for each t∈R⁺, where *n*∈*N*. The collection of such functions will be denoted by $P_wL(R^+,X)$.

Remark 2.1: The function *g* is measurable but not necessarily continuous. In the following, we describe some of theproperties of the *w*-periodic limit function.

Proposition 2.1: If f, f_1 and f_2 are *w*-periodic limit and $g(t) = \lim_{t\to\infty} f(t+mw)$ is well defined for t∈R⁺, then the following statements are true: [(a)]

1. f_1+f_2 is *w*-periodic limit,
2. *cf* is *w*-periodic limit for every scalar *c*,
3. *g*(*t*+*w*)=*g*(*t*) for each *t*∈*R*⁺,
4. *g* is bounded on *R*⁺; moreover $\| g \|_\infty \leq \| f \|_\infty$,

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5. $f_a(t)=f(t+a)$ is *w*-periodic limit for each fixed $a\in R^+$.

Proposition 2.2: $AP_w(R^+,X)$ is a Banach space.

Proposition 2.3: $P_wL(R^+,X)$ is a Banach space.

Proposition 2.4: Let $f\in P_wL(R^+,X)$ and $g(t)=\lim_{n\to\infty}f(t+nw)$ be well defined for each $t\in R^+$. If $g(t)=\lim_{n\to\infty}f(t+nw)$ is uniformly on R^+, then $f\in AP_w(R^+,X)$.

Proposition 2.5: Let $\phi:X\to Y$ be a function which is uniformly continuous on the bounded subsets of X and such that ϕ maps bounded subsets of X into bounded subsets of Y. Then for all $f\in P_wL(R^+,X)$, the composition Theorem $\varphi\circ f:=[t\to\varphi(f(t))]\in P_wL(R^+,X)$.

Proposition 2.6: Let $(X,||.||)$ be a Banach space over the field K where $K=R$ or C. If $a(t)\in P_wL(R^+,R)$ and $f(t)\in P_wL(R^+,X)$ then $a(t)f(t)\in P_wL(R^+,X)$.

Lemma 2.1: Let $f\in P_wL(R^+,X)$ and $(t_n)_{n\in N}$ be a sequence with $t_n\to\infty$ as $n\to\infty$. We denote $f_{t_n}:[0,\infty)\to X$ defined by $f_{t_n}(t)=f(t+t_n)$. We assume that $f_{t_n}\to F$ uniformly on compact subsets of $[0,\infty)$, then $F\in P_wL(R^+,X)$.

Proof: It is clear that F is continuous. The function f is *w*-periodic limit function so we have $g(t)=\lim_{n\to\infty}f(t+nw)$ well defined for each $t\in R^+$ [7-10].

$$\lim_{m\to\infty}f_{t_n}(t+mw)=\lim_{m\to\infty}f(t+t_n+mw)=g(t+t_n)=g_{t_n}(t),t\geq 0.$$

For $\varepsilon>0$ given, we select $n_0\in N$ such that:

$$\|F(s)-f(s+t_n)\|<\varepsilon 2\qquad s\in[t,t+w],$$

$$\|f(t+t_n+mw)-g_{t_n}(t)\|<\varepsilon 2,\qquad t\geq 0,$$

for every $m\geq n_0$. Hence for $m\geq n_0$, we have

$$\|F(t+mw)-g_{t_n}(t)\|\leq\|F(t+mw)-f(t+t_n+mw)$$

$$\|+\|f(t+t_n+mw)-g_{t_n}(t)\|$$

$$\leq\varepsilon 2+\varepsilon 2=\varepsilon.$$

We denote $G(t)=g_{t_n}(t)=g(t+t_n)$, and $G(t)$ well defined for each $t\in R^+$, so that $\lim_{m\to\infty}F(t+mw)=g_{t_n}(t)=G(t)$ well defined for each $t\in R^+$, which implies that $F\in P_wL(R^+,X)$.

The following definition is an alternating *w*-periodic limit function in two dimensions.

Definition 2.3

A jointly continuous function $f:R^+\times X\to X$ is *w*-periodic limit in $t\in R^+$ uniformly for x in bounded subsets of X if for every bounded subset K of X, $\{f(t,x):t\in R^+,x\in K\}$ is bounded and $\lim_{n\to\infty}f(t+nw,x)=g(t,x)$ exists for each $t\in R^+$ and each $x\in K$. The collection of such function will denoted by $P_wL(R^+\times X,X)$.

Proposition 2.7 [13]. If $f:R^+\times X\to X$ is *w*-periodic limit in $t\in R^+$ uniformly for x in bounded subsets of X and f satisfies a Lipschitz condition in x uniformly in $t\in R^+$, then g satisfies the same Lipschitz condition in x uniformly in t.

Proposition 2.8: Let $f:R^+\times X\to X$ be *w*-periodic limit in $t\in R^+$ uniformly for x in bounded subsets of X and assume that f satisfies a Lipschitz condition in x uniformly in $t\in R^+$:

$$\|f(t,x)-f(t,y)\|\leq L\|x-y\|,$$

for all $x,y\in X$ and $t\in R^+$, where L is a positive constant. Let $\phi:R^+\to X$ be *w*-periodic limit, then the function $F:R^+\to X$ defined by $f(t)=f(t,(t))$ is *w*-periodic limit.

Existence of Asymptotically *w*-Periodic Solutions of the Fractional Semilinear Problem

In this section, we first introduce more accurate conditions for problem (1) and then we investigate the existence of asymptotically *w*-periodic solution. We consider the fractional semilinear problem (1) and $A(t):D(A)\subset X\to X$ is a generator an evolution family $(U(t))_{t\geq s\geq 0}$ on X and $(U(t))_{t\geq s\geq 0}$ satisfying [(a)] [11-14]:

1. $U(t,t)=I$ for all $t\in R$, where I is the identity operator,
2. $U(t,s)U(s,r)=U(t,r)$ for all $t\geq s\geq r$,
3. The map $(t,s)\mapsto U(t,s)x$ is continuous for every fixed $x\in X$.

We consider the following hypothesis:

(H_1) Evolution family $(U(t,s))_{t\geq s\geq 0}$ is a uniformly continuous, such that:

$$U(t+w,s+w)=U(t,s)\quad for\ all\quad t\geq s,$$

there exist M>0 and >0 such that:

$$\|U(t,s)\|\leq M\exp^{-\delta(t-s)}|t-s|^{(1-\alpha)}\quad for\quad t>s\geq 0.$$

(H_2) The function $f:R^+\times X\to X$ is *w*-periodic limit in $t\in R^+$ uniformly for x in bounded subsets of X and f satisfies a Lipschitz condition in x uniformly in $t\in R^+$:

$$\|f(t,x)-f(t,y)\|\leq L\|x-y\|,$$

for all $x,y\in X$ and $t\in R^+$, where L is positive constant.

Definition 3.1

A continuous function $x:R^+\to X$ is said to be a mild solution of (1), if x to

$$x(t)=U(t,0)x_0+1\Gamma(\alpha)\int_0^t U(t,s)(t-s)^{1-\alpha}f(s,x(s))ds.$$

Lemma 3.1: We assume that H_1 is satisfied and that $f\in P_wL(R^+\times X,X)$ then

$v(t)=\int_0^t U(t,s)(t-s)^{1-\alpha}f(s,x(s))ds$, is in $AP_w(R^+,X)$ for $t\in R^+$.

Proof: We denote $F(s)=f(s,x(s))$. In view of Proposition 2.8, if $x\in P_wL(R^+,X)$ then $F\in P_wL(R^+,X)$. So

$$\lim_{n\to\infty}F(t+nw)=g(t),$$

is well defined for each $t\in R^+$. So the Proposition 2.1, there exists also a positive constant k, so that $\|g\|_\infty\leq\|F\|_\infty\leq k$ and $g(t)=g(t+w)$. We have that:

$$v(t+nw)=\int_0^{t+nw}U(t+nw,s)(t+nw-s)^{1-\alpha}F(s)ds$$

$$=\int_{-nw}^{t}U(t+nw,s+nw)(t+nw-s-nw)^{1-\alpha}F(s+nw)ds$$

$$=\int_{-nw}^{t}U(t,s)(t-s)^{1-\alpha}F(s+nw)ds$$

$$=\int_{-nw}^{0}U(t,s)(t-s)^{1-\alpha}F(s+nw)ds+\int_0^t U(t,s)(t-s)^{1-\alpha}F(s+nw)ds$$

$$=I_1(t,n)+I_2(t,n).$$

Next we will prove that $I_1(t,n)$ is a Cauchy sequence in X for each $t\in R^+$. Let >0. For any $p\in N$, we observe that

$$I_1(t,n+p)-I_1(t,n)=\int_{-(n+p)w}^{0}U(t,s)(t-s)^{1-\alpha}F(s+(n+p)w)ds$$
$$-\int_{-nw}^{0}U(t,s)(t-s)^{1-\alpha}F(s+nw)ds$$
$$=\int_{-(n+p)w}^{-nw}U(t,s)(t-s)^{1-\alpha}F(s+(n+p)w)ds$$
$$+\int_{-nw}^{0}U(t,s)(t-s)^{1-\alpha}(F(s+(n+p)w)-F(s+nw))ds$$
$$=I_3(t,n,p)+I_4(t,n,p).$$

Now, we consider the term $I_3(t,n,p)$

$$\| I_3(t,n,p)\|\le \int_{-(n+p)w}^{-nw}\| U(t,s)\|\| t-s|^{1-\alpha}\| F(s+(n+p)w)\| ds$$
$$\le \int_{-(n+p)w}^{-nw}M\exp^{-\delta(t-s)}|t-s|^{1-\alpha}|t-s|^{1-\alpha}\; kds$$
$$\le kM\int_{nw}^{(n+p)w}\exp^{-\delta s}ds$$
$$\le kM\delta\exp^{-\delta nw}.$$

We can choose $N_1 \in N$ such that $kM\delta\exp^{-\delta nw}\le\varepsilon$ when $n \ge N_1$. Therefore $\| I_3(t,n,p)\|\le\varepsilon$, whenever $n \ge N_1$ is uniformly for $t\in R^+$.

For $n \ge N_1$, we consider $I_4(t,n,p)$

$$I_4(t,n,p)=\int_{-N_1w}^{0}U(t,s)(t-s)^{1-\alpha}(F(s+(n+p)w)-F(s+nw))ds$$
$$+\int_{-nw}^{-N_1w}U(t,s)(t-s)^{1-\alpha}(F(s+(n+p)w)-F(s+nw))ds$$
$$=I_5(t,n,p)+I_6(t,n,p).$$

We consider the term $I_5(t,n,p)$

$$\| I_5(t,n,p)\|\le \int_{-N_1w}^{0}\| U(t,s)\|\| t-s|^{1-\alpha}\| F(s+(n+p)w)-F(s+nw)\| ds$$
$$\le \int_{-N_1w}^{0}M\exp^{-\delta(t-s)}|t-s|^{1-\alpha}|t-s|^{1-\alpha}\| F(s+(n+p)w)-F(s+nw)\| ds$$
$$=\int_{0}^{N_1w}M\exp^{-\delta(t+s)}\| F(-s+(n+p)w)-F(-s+nw)\| ds$$
$$=\int_{0}^{N_1w}M\exp^{-\delta(t+N_1w-s)}\| F(s+(n-N_1+p)w)-F(s+(n-N_1)w)\| ds$$
$$\le \int_{0}^{N_1w}M\exp^{-\delta(N_1w-s)}\| F(s+(n-N_1+p)w)-F(s+(n-N_1)w)\| ds$$
$$\le \int_{0}^{N_1w}M\exp^{-\delta(N_1w-s)}\| F(s+(n-N_1+p)w)-g(s)\| ds$$
$$+\int_{0}^{N_1w}M\exp^{-\delta(N_1w-s)}\| F(s+(n-N_1)w)-g(s)\| ds,$$

for each $s\in[0,N_1,w]$, we have:

$$M\exp^{-\delta(N_1w-s)}\| F(s+(n-N_1+p)w)-g(s)\|\le 2Mk\exp^{-\delta(N_1w-s)},$$

and

$$\int_{0}^{N_1w}2Mk\exp^{-\delta(N_1w-s)}ds=2Mk\delta(1-\exp^{-\delta N_1w}),$$

since $f\in P_wL(R^+,X)$, for each $s\in[0,N_1,w]$, we have

$$\lim_{n\to\infty}M\exp^{-\delta(N_1w-s)}\| F(s+(n-N_1)w)-g(s)\|=0,$$

we have $\| F(s+(n-N_1)w)-g(s)\|\le 2k$, so by the Lebegue's Dominated Convergence Theorem, we deduce that:

$$\lim\int_{0}^{N_1w}M\exp^{-\delta(N_1w-s)}\| F(s+(n-N_1)w)-g(s)\| ds=0.$$

Also, we have

$$\lim\int_{0}^{N_1w}M\exp^{-\delta(N_1w-s)}\| F(s+(n-N_1+p)w)-g(s)\| ds=0.$$

Therefore, we can select $N_2\in N$ $(N_2\ge N_1)$ such that $\|I_5(t,n,p)\|\le\varepsilon$ whenever $n\ge N_2$ uniformly for $t\in R^+$.

Now we consider the term $I_6(t,n,p)$

$$\| I_6(t,n,p)\|\le \int_{-nw}^{-N_1w}\| U(t,s)\|\| t-s|^{1-\alpha}\| F(s+(n+p)w)-F(s+nw)\| ds$$
$$\le 2kM\int_{-nw}^{-N_1w}\exp^{-\delta(t-s)}ds$$
$$\le 2kM\int_{N_1w}^{nw}\exp^{-\delta(t+s)}ds$$
$$\le 2kM\int_{N_1w}^{\infty}\exp^{-\delta(t+s)}ds$$
$$\le 2kM\delta\exp^{-\delta N_1w}\le 2\varepsilon,$$

uniformly for $t\in R^+$. Since

$$\| I_1(t,n+p)-I_1(t,n)\|\le\| I_3(t,n,p)\|+\| I_5(t,n,p)\|+\| I_6(t,n,p)\|,$$

we deduce that

$$\| I_1(t,n+p)-I_1(t,n)\|\le 4\varepsilon,$$

when $n \ge N_2$. Therefore $I_1(t,n)$ is a Cuachy sequence. So we can denote $h(t)=\lim_{n\to\infty}I_1(t,n)$ for each $t\in R^+$. Note also $h(t)=\lim_{n\to\infty}I_1(t,n)$ uniformly for $t\in R^+$.

Now consider the term $I_2(t,n)$. Since g is measurable, $\int U(t,s)(t-s)^{1-\alpha}g(s)ds$ is well defined for each $t\in R^+$.

For $mw\le t\le(m+1)w$, $m\in N$, we have

$$\| I_2(t,n)-\int_0^t U(t,s)(t-s)^{1-\alpha}g(s)ds\|\le\int_0^t\| U(t,s)\|\| t-s|^{1-\alpha}\| F(s+nw)-g(s)\| ds$$
$$\le\int_0^t M\exp^{-\delta(t-s)}|t-s|^{1-\alpha}|t-s|^{1-\alpha}\| F(s+nw)-g(s)\| ds$$
$$=\int_0^{mw}M\exp^{-\delta(t-s)}\| F(s+nw)-g(s)\| ds$$
$$+\int_{mw}^{t}\exp^{-\delta(t-s)}\| F(s+nw)-g(s)\| ds$$
$$\le M\sum_{k=0}^{m-1}\int_{kw}^{(k+1)w}\exp^{-\delta(t-s)}\| F(s+nw)-g(s)\| ds$$
$$+M\int_{mw}^{(m+1)w}\exp^{-\delta(t-s)}\| F(s+nw)-g(s)\| ds$$
$$=M\sum_{k=0}^{m}\int_{kw}^{(k+1)w}\exp^{-\delta(t-s)}\| F(s+nw)-g(s)\| ds$$

For each $s\in[0,w]$, we have $\lim_{n\to\infty}\| F(s+nw)-g(s)\| ds=0$ and $\| F(s+nw)-g(s)\|\le 2k$. By Lebesgue's Dominated convergence Theorem, we obtain

$$\lim_{n\to\infty}\int_0^w\| F(s+nw)-g(s)\| ds=0.$$

For $\varepsilon>0$. There exists $N_3\in N$ such that:

$$\int_0^w\| F(s+nw)-g(s)\| ds\le\varepsilon,$$

when $n \ge N_3$. For any $i\in N$, we have:

$$\int_{iw}^{(i+1)w}\| F(s+nw)-g(s)\| ds=\int_0^w\| F(s+iw+nw)-g(s+iw)\| ds$$
$$=\int_0^w\| F(s+iw+nw)-g(s)\| ds\le\varepsilon,$$

when $n \ge N_3$. Therefore

$$\| I_2(t,n)-\int_0^t U(t,s)(t-s)^{1-\alpha}g(s)ds\|\le M\sum_{k=0}^{m}\exp^{-\delta(t-(k+1)w)}\varepsilon$$
$$\le M\varepsilon 1-exp^{-sw},$$

when $n \ge N_3$ uniformly for $t\in R^+$. Therfore:

$$\lim_{n\to\infty} I_2(t,n) = \int_0^t U(t,s)(t-s)^{1-\alpha} g(s)ds,$$

Now we have:

$$\lim_{n\to\infty} v(t+nw) = \lim_{n\to\infty} I_1(t,n) + \lim_{n\to\infty} I_2(t,n)$$
$$= h(t) + \int_0^t U(t,s)(t-s)^{1-\alpha} g(s)ds,$$

uniformly for $t\in R^+$. by Proposition 2.4, we get $v \in AP_w(R^+, X)$.

Theorem 3.2: We assume the hypothesis (H_1) and (H_2) are satisfied. If $ML < \Gamma(\alpha)\delta$, then there exists a unique asymptotically *w*-periodic mild solution of problem (1).

Proof: We define the nonlinear operator ∧ by the expression:

$$(\Lambda\varphi)(t) = U(t,0)x_0 + 1\Gamma(\alpha)\int_0^t U(t,s)(t-s)^{1-\alpha} f(s,\varphi(s))ds$$
$$= U(t,0)x_0 + 1\Gamma(\alpha)(\Psi\varphi)(t),$$

where

$$(\Psi\varphi)(t) = \int_0^t U(t,s)(t-s)^{1-\alpha} f(s,\varphi(s))ds.$$

According to the hypothesis

$$U(t+w, 0+w) = U(t,0),$$

so $U(t,0)x_0 \in P_w(R^+, X) \subseteq AP_w(R^+, X)$.

According to the lemma 3.1 the operator ψ maps $AP_w(R^+,X)$ into itself, therefore the operator ∧ maps $AP_w(R^+,X)$ into itself.

We have

$$\|(\Lambda\varphi)(t) - (\Lambda\psi)(t)\| = 1\Gamma(\alpha)\| \int_0^t U(t,s)(t-s)^{1-\alpha}(f(s,\varphi(s)) - f(s,\psi(s)))ds \|$$
$$\le 1\Gamma(\alpha)\int_0^t \|U(t,s)\| |t-s|^{1-\alpha} \|f(s,\varphi(s)) - f(s,\psi(s))\| ds$$
$$\le L\Gamma(\alpha)\int_0^t \|U(t,s)\| |t-s|^{1-\alpha} \|\varphi(s) - \psi(s)\| ds$$
$$\le LM\Gamma(\alpha)\int_0^t \exp^{-\delta(t-s)} \|\varphi(s) - \psi(s)\| ds$$
$$\le LM\Gamma(\alpha)\int_0^t \exp^{-\delta(t-s)} \|\varphi - \psi\|_\infty$$
$$\le LM\Gamma(\alpha)(1-\exp^{-\delta t})\delta \|\varphi - \psi\|_\infty$$
$$\le LM\Gamma(\alpha)\delta \|\varphi - \psi\|_\infty,$$

hence we have

$$\|\Lambda\varphi - \Lambda\psi\|_\infty \le LM\Gamma(\alpha)\delta \|\varphi - \psi\|_\infty .$$

Which proves that ∧ is a contraction and we conclude that ∧ has a unique fixed point in $AP_w(R^+,X)$. The proof is complete.

Existence of Asymptotically *w*-periodic Solutions of the Fractional Semilinear Integro-differential Equation

In this section, we examine the existence of a solution for problem (2) by generalizing the items raised in the previous section. We consider the fractional semilinear integro-differential eqn. (2) that X is separable Banach space.

By generalizing the definition of $P_w L(R^+\times X,X)$ to three dimensions, we have the following definition.

Definition 4.1

A continuous function $f:R^+\times X\times X\to X$ is *w*-periodic limit in $t\in R^+$ uniformly for (x,y) in bounded subsets of $X\times X$ if for every bounded subset K of $X\times X$, $\{f(t,x,y):t\in R^+,(x,y)\in K\}$ is bounded and $\lim_{n\to\infty} f(t+nw,x,y) = g(t,x,y)$ exists for each $t\in R^+$ and each $(x,y)\in K$ the collection of such functions will be denoted by $P_w L(R^+\times X\times X,X)$

Now, we assume that the following hypothesis satisfy

(H_3) $f(t,x,y):R^+\times X\times X\to X$ satisfies the caratheodory type conditions, i.e, $f(x,y)$ is measurable for $(x,y)\in X\times X$ and $f(t,.,.)$ is continuous for a.e, $t\ge 0$ and f is *w*-periodic limit function in $t\in R^+$ uniformly for (x,y) in bounded subsets of $X\times X$.

(H_4) $u(t,s,x):R^+\times R^+X\to X$ is continuous. u is *w*-periodic limit and $\lim_{n\to\infty} u(t+nw,s,x)ds = v(t,s,x)$ uniformly in $t\in R^+$ and the function $\psi(t)$ with definition $\Psi(t) = \int_0^t u(t,s,x(s))ds$ is bounded.

Remark 4.1: u is *w*-periodic limit so $\lim_{n\to\infty} u(t+nw,s,x)ds = v(t,s,x)$ exists and uniformly for each $t\in R^+$, so we have

$$\lim_{n\to\infty}\int_0^t u(t+nw,s,x)ds = \int_0^t \lim_{n\to\infty} u(t+nw,s,x)ds = \int_0^t v(t,s,x)ds.$$

Since $v(t,s.x)$ is measurable, $\int_0^t v(t,s,x)ds$ well defined for each $t\in R^+$. Therefore $\int_0^t u(t,s,x)ds$ is *w*-periodic limit.

Definition 4.2

A continuous function $x:R^+\to X$ is said to be a mild solution of (2), if x satisfies to

$$x(t) = U(t,0)x_0 + 1\Gamma(\alpha)\int_0^t U(t,s)(t-s)^{1-\alpha} f(s,x(s),\int_0^t u(t,s,x(s))ds)ds.$$

With the generalization of the Proposition 2.8 to three dimensions, we have the following proposition.

Proposition 4.1: We assume that and assume the hypothesis (H_3) is satisfied and f satisfies a

$$\| f(t,x_1,y_1) - f(t,x_2,y_2)\| \le L(\|(x_1,y_1) - (x_2,y_2)\|),$$

for all $(x_1,y_1),(x_2,y_2)\in X\times X$, $t\in R^+$, where L is a positive constant. Let $\phi:R^+\to X$ and $\psi:R^+\to X$ w-periodic limit, then the function $F:R^+\to X$ defined by $F(t)=f(t,\phi(t),\psi(t))$ is w-periodic limit.

Proof: Since ϕ and are w -periodic limit functions, we have

$$\lim_{n\to\infty} \varphi(t+nw) = \Phi(t), \tag{3}$$
$$\lim_{n\to\infty} \psi(t+nw) = \Psi(t), \tag{4}$$

for each $t\in R^+$. On the other hand, we have

$$\lim_{n\to\infty} f(t+nw,x,y) = g(t,x,y), \tag{5}$$

for each $t\in R^+$ and each $(x,y)\in K$.

$\phi(t)$ and $\psi(t)$ are bounded and by Proposition 2.1(d), $\varphi(t)$ and $\psi(t)$ are bounded, so we can choose a bounded subset K of X such that $\phi(t)$, $\varphi(t)$, $\psi(t)$ and $\psi(t)\in K$ for all $t\in R^+$. Thus $F(t)$ is bounded.

Let us consider the function $G(t)=g(t,\varphi(t),\psi(t))$. Note that

$$\|F(t+nw) - G(t)\| \le \| f(t+nw,\varphi(t+nw),\psi(t+nw)) - f(t+nw,\Phi(t),\Psi(t))\|$$
$$+ \| f(t+nw,\Phi(t),\Psi(t)) - g(t,\Phi(t),\Psi(t))\|$$
$$\le L\|(\varphi(t+nw),\psi(t+nw)) - (\Phi(t),\Psi(t))\|$$
$$+ \| f(t+nw,\Phi(t),\Psi(t)) - g(t,\Phi(t),\Psi(t))\|,$$

we deduce from eqns. (3)-(5):

$$\lim_{n\to\infty} F(t+nw) = G(t),$$

for each $t\in R^+$, finished the proof.

The following result is a generalization from Lemma 3.1.

Corollary 4.1. We assume that H_1 is satisfied and $f\in P_w L(R^+\times X\times X,X)$ then

$$v(t) = \int_0^t U(t,s)(t-s)^{1-\alpha} f(s, x(s), y(s))ds,$$

is in $AP_w(R^+,X)$ for t∈R⁺.

Proof: we denote F(s)=$f(t,x(t),y(s))$. In view Proposition 4.1 if $x,\in P_w L(R^+,X)$ then $F \in P_w L(R^+, X)$. So we can provide the proof of Lemma 3.1 for the new F.

Proposition 4.2: If the Banach space X is separable. Assume that the hypotheses H_1 and H_3 are satisfied then for each $x_0 \in X$, the problem (2) has at least one mild solution x in $C(R^+,X)$.

Using Corollary 4.1 and the generalization of Theorem 3.2, we have the following theorem.

Theorem 4.3: We assume that X is separable Banach space and the hypothesis (H_1),(H_3) and (H_4) are satisfied. If $2mML < \Gamma(\alpha)\delta$, that $\int_0^t u(t,s,x(s))ds \le m$ then problemin eqn.(2) has a asymptotically w-periodic mild solution.

Proof: By distribution of Theorem (3.2), we have

$$(\Lambda\varphi)(t) = U(t,0)x_0 + 1\Gamma(\alpha)\int_0^t U(t,s)(t-s)^{1-\alpha} f(s,\varphi(s), \int_0^s u(s,\tau,\varphi(\tau))d\tau)ds$$
$$= U(t,0)x_0 + 1\Gamma(\alpha)(\Psi\varphi)(t),$$

where

$$(\Psi\varphi)(t) = \int_0^t U(t,s)(t-s)^{1-\alpha} f(s,\varphi(s), \int_0^s u(s,\tau,\varphi(\tau))d\tau)ds,$$

and $f(t,\varphi(t),\psi(T)) \in P_w L(R^+, X)$. So according to the Corollary 4.1 the operator ψ maps $AP_w(R^+,X)$ into itself. Therefore the operator ∧ maps $AP_w(R^+,X)$ into itself.

We denote $\int_0^t u(t,s,\varphi(s))ds = I(\varphi(s))$ for short. We have

$$\|(\Lambda\varphi)(t) - (\Lambda\psi)(t)\| = \| 1\Gamma(\alpha)\int_0^t U(t,s)(t-s)^{1-\alpha}(f(s,\varphi(s),I(\varphi(s))) - f(s,\psi(s),I(\psi(s)))ds\|$$
$$\le 1\Gamma(\alpha)\int_0^t \|U(t,s)\| |t-s|^{1-\alpha} \| f(s,\varphi(s),I(\varphi(s))) - f(s,\psi(s),I(\psi(s)))\| ds$$
$$\le L\Gamma(\alpha)\int_0^t \|U(t,s)\| |t-s|^{1-\alpha} \|(\varphi(s),I(\varphi(s))) - (\psi(s),I(\psi(s)))\| ds$$
$$\le LM\Gamma(\alpha)\int_0^t \exp^{-\delta(t-s)} \|(\varphi(s)-\psi(s), I(\varphi(s)) - I(\psi(s)))\| ds$$
$$\le LM\Gamma(\alpha)\int_0^t \exp^{-\delta(t-s)} \|(\varphi-\psi, I(\varphi)-I(\psi))\|_\infty ds$$
$$\le LM\Gamma(\alpha)(1-\exp^{-\delta t})\delta \|(\varphi-\psi, I(\varphi)-I(\psi))\|_\infty$$
$$\le LM\Gamma(\alpha)\delta \|(\varphi-\psi, I(\varphi)-I(\psi))\|_\infty .$$
$$\le LM\Gamma(\alpha)\delta \|\varphi-\psi\|_\infty \|I(\varphi)-I(\psi)\|_\infty .$$
$$\le 2mLM\Gamma(\alpha)\delta \|\varphi-\psi\|_\infty$$

Hence we have

$$\|\Lambda\varphi - \Lambda\psi\|_\infty \le 2mLM\Gamma(\alpha)\delta \|\varphi-\psi\|_\infty,$$

which proves that ∧ is a contraction and we conclude that ∧ has a unique fixed point in $AP_w(R^+,X)$. the proof is complete.

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