Journal of Generalized Lie Theory and Applications Vol. 3 (2009), No. 3, 183–190

On anti-structurable algebras and extended Dynkin diagrams

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Abstract

We construct Lie superalgebras $\mathfrak{osp}(2n+1 \mid 4n+2)$ and $\mathfrak{osp}(2n \mid 4n)$ starting with certain classes of anti-structurable algebras via the standard embedding Lie superalgebra construction corresponding to (ϵ, δ)-Freudenthal Kantor triple systems.

AMS MSC 2000: 17A30, 17B60

1 Introduction

1.1 (ϵ, δ)-Freudenthal Kantor triple systems, δ-Lie triple systems, and Lie (super)algebras

We are concerned in this paper with triple systems which have finite dimension over a field Φ of characteristic $\neq 2$ or 3, unless otherwise specified.

In order to render this paper as self-contained as possible, we recall first the definition of a generalized Jordan triple system of second order (for short GJTS of 2nd order).

Definition 1.1. A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a GJTS of 2nd order if the following conditions are fulfilled:

$$\big(ab(xyz)\big) = \big((abx)yz\big) - \big(x(bay)z\big) + \big(xy(abz)\big) \tag{1.1}$$

$$K\big(K(a,b)x, y\big) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0 \tag{1.2}$$

where

$$L(a,b)c := (abc) \quad \text{and} \quad K(a,b)c := (acb) - (bca)$$

Definition 1.2. A Jordan triple system (for short JTS) satisfies (1.1) and $(abc) = (cba)$, $\forall a, b, c \in V$.

We can generalize the concept of GJTS of the 2nd order as follows (see [10, 11, 13, 15, 32]).

Definition 1.3. For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$\big(ab(xyz)\big) = \big((abx)yz\big) + \varepsilon\big(x(bay)z\big) + \big(xy(abz)\big) \tag{1.3}$$

$$K\big(K(a,b)x, y\big) - L(y,x)K(a,b) + \varepsilon K(a,b)L(x,y) = 0 \tag{1.4}$$

where

$$L(a,b)c := (abc), \quad K(a,b)c := (acb) - \delta(bca) \tag{1.5}$$

is called an (ε, δ)-Freudenthal-Kantor triple system (for short (ε, δ)-FKTS).

Remark 1.4. Note that $K(b,a) = -\delta K(a,b)$.

Definition 1.5. An (ε,δ)-FKTS U is called unitary if the identity map Id is contained in $\kappa := K(U,U)$, i.e., if there exist $a_i, b_i \in U$ such that $\Sigma_i K(a_i,b_i) = Id$.

Let U be an (ε,δ)-FKTS and let $V_k, k = 1,2,3$, be subspaces of U. We denote by (V_1,V_2,V_3) the subspace of U spanned by elements (x_1,x_2,x_3), $x_k \in V_k, k = 1,2,3$.

Definition 1.6. A subspace V of U is called an ideal of an (ε,δ)-FKTS U if the following relations hold: $(V,U,U) \subseteq V$, $(U,V,U) \subseteq V$, $(U,U,V) \subseteq V$. U is called simple if $(\ ,\ ,\ )$ is not a zero map and U has no nontrivial ideal.

We denote the triple products by (xyz), $\{xyz\}$, $[xyz]$, and $\langle xyz\rangle$ upon their suitability.

Remark 1.7. We note that the concept of GJTS of 2nd order coincides with that of $(-1,1)$-FKTS. Thus we can construct the simple Lie algebras by means of the standard embedding method (see [5, 10, 11, 12, 13, 15, 16, 17, 22, 32]).

Remark 1.8. We note that the two pairs of identities (1.3-1.4) and (1.6) are equivalent

$$\big[L(a,b),L(x,y)\big] = L\big((abx),y\big) + \varepsilon L\big(x,(bay)\big) \tag{1.6a}$$

$$K\big(K(a,b)x,y\big) - K\big((yxa),b\big) - K\big(a,(yxb)\big) = 0 \tag{1.6b}$$

where $\varepsilon = \pm 1, \delta = \pm 1$ and $L(a,b), K(a,b)$ are defined by (1.5).

Indeed, from (1.3) and (1.4) follows (1.6b). Conversely, from (1.6a) and (1.6b) it follows that (1.4) holds.

For an (ε,δ)-FKTS U, we denote

$$S(a,b) := L(a,b) + \varepsilon L(b,a), \quad A(a,b) := L(a,b) - \varepsilon L(b,a)$$

where $L(a,b)$ is defined by (1.5).

Remark 1.9. We note that $S(a,b) = \varepsilon S(b,a)$.

Then $S(a,b)$ (resp., $A(a,b)$) is a derivation (resp., anti-derivation) of U.

Indeed, we note that the identities (1.7) and (1.8) are valid.

$$\big[S(a,b),L(c,d)\big] = L\big(S(a,b)c,d\big) + L\big(c,S(a,b)d\big) \tag{1.7}$$

$$\big[A(a,b),L(c,d)\big] = L(A(a,b)c,d) - L\big(c,A(a,b)d\big) \tag{1.8}$$

Definition 1.10. For $\delta = \pm 1$, a triple system $(a,b,c) \mapsto [abc], a,b,c \in V$ is called a δ-Lie triple system (for short δ-LTS) if the following identities are fulfilled:

$$[abc] = -\delta[bac]$$

$$[abc] + [bca] + [cab] = 0$$

$$\big[ab[xyz]\big] = \big[[abx]yz\big] + \big[x[aby]z\big] + \big[xy[abz]\big]$$

where $a,b,x,y,z \in V$. An 1-LTS is a LTS, while a -1-LTS is called an anti-LTS, by [11].

Proposition 1.11 (see [11, 15]). *Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J\langle xyz\rangle = \langle JxJyJz\rangle$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta),[xyz])$ is an LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product (1.9):*

$$[xyz] := \langle xJyz\rangle - \delta\langle yJxz\rangle + \delta\langle xJzy\rangle - \langle yJzx\rangle \tag{1.9}$$

Corollary 1.12. *Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes an LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product (1.10):*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon\big(L(d,a) - \delta L(b,c)\big) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} \tag{1.10}$$

Remark 1.13. Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(\varepsilon,\delta) = D(T(\varepsilon,\delta),T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated to $T(\varepsilon,\delta)$, where $D(T(\varepsilon,\delta),T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$, i.e.,

$$D\big(T(\varepsilon,\delta),T(\varepsilon,\delta)\big) := \left\{ \begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix} \right\}_{\text{span}}$$

$$T(\varepsilon,\delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\Big|\; x,y \in U(\varepsilon,\delta) \right\}_{\text{span}}$$

Remark 1.14. $L(\varepsilon,\delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ is the 5-graded Lie (super)algebra such that $L_{-1} \oplus L_1 = T(\varepsilon,\delta)$, $D(T(\varepsilon,\delta),\, T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_2$, and $[L_i, L_j] \subseteq L_{i+j}$. This Lie (super)algebra construction is one of the reasons to study nonassociative algebras and triple systems.

1.2 δ-structurable algebras

The existence of the class of nonassociative algebras called structurable algebras is an important generalization of Jordan algebras giving a construction of Lie algebras. Hence from our concept, by means of triple products, we define a generalization of such class to construct Lie superalgebras as well as Lie algebras. Our start point briefly described in a historical setting is the construction of Lie (super)algebras starting from a class of nonassociative algebras. Hence within the general framework of (ϵ,δ)-FKTSs $(\epsilon,\delta = \pm 1)$ and the standard embedding Lie (super)algebra construction studied in [5, 6, 10, 11, 12, 17] (see also references therein) we defined δ-structurable algebras (see [18]) as a class of nonassociative algebras with involution which coincides with the class of structurable algebras for $\delta = 1$ as introduced and studied in [1, 2]. Structurable algebras are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to GJTSs of 2nd order, or $(-1,1)$-FKTSs, as introduced and studied in [20, 21] and further studied in [3, 4, 19, 26, 27, 28, 29, 30] (see also references therein). Their importance lies with constructions of 5-graded Lie algebras $L(\epsilon,\delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, $[L_i, L_j] \subseteq L_{i+j}$. For $\delta = -1$, the anti-structurable algebras (see [18]) are a class of nonassociative algebras that may similarly shed light on the notion of $(-1,-1)$-FKTSs hence, by [5, 6], on the construction of Lie superalgebras and Jordan algebras as it will be shown.

Throughout the paper, it is assumed that $(\mathcal{A},^{-})$ is a finite-dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e., $\overline{\overline{x}} = x$ and $\overline{xy} = \overline{y}\,\overline{x}$ for $x,y \in \mathcal{A}$) over Φ. The identity element of $\mathcal{A}$ is denoted by 1.

Remark 1.15. By [1] we have $\mathcal{A}=\mathcal{H}\oplus\mathcal{S}$, where $\mathcal{H}=\{a\in\mathcal{A} \mid \overline{a}=a\}$ and $\mathcal{S}=\{a\in\mathcal{A} \mid \overline{a}=-a\}$.

Suppose $x,y,z \in \mathcal{A}$. Put $[x,y] := xy - yx$, $[x,y,z] := (xy)z - x(yz)$. Note that (1.11) is valid.

$$\overline{[x,y,z]} = -[\overline{z},\overline{y},\overline{x}] \tag{1.11}$$

Let L_x, R_x be defined by $L_x(y) := xy, R_x(y) := yx, x,y \in \mathcal{A}$. For $\delta = \pm 1$, define (1.12) and (1.13).

$$^{\delta}V_{x,y} := L_{L_x(\overline{y})} + \delta\big(R_xR_{\overline{y}} - R_yR_{\overline{x}}\big) \tag{1.12}$$

$$^{\delta}B_{\mathcal{A}}(x,y,z) :=^{\delta} V_{x,y}(z) = (x\overline{y})z + \delta\big[(z\overline{y})x - (z\overline{x})y\big], \quad x,y,z \in \mathcal{A} \tag{1.13}$$

Definition 1.16. $^{+}B_{\mathcal{A}}(x,y,z)$ is called the triple system obtained from the algebra $(\mathcal{A},^{-})$. We call $^{-}B_{\mathcal{A}}(x,y,z)$ the anti-triple system obtained from the algebra $(\mathcal{A},^{-})$.

We will write for short $V_{x,y} :=^{\delta} V_{x,y}$, $B_{\mathcal{A}} := (^{\delta}B_{\mathcal{A}}, \mathcal{A})$.

Remark 1.17. The upper left index notation is chosen in order not to be mixed with the upper right index notation of [1] which has a different meaning.

Definition 1.18. A unital nonassociative algebra with involution $(\mathcal{A},^{-})$ is called a structurable algebra if the following identity is fulfilled:

$$\big[V_{u,v}, V_{x,y}\big] = V_{V_{u,v}(x),y} - V_{x,V_{v,u}(y)} \tag{1.14}$$

for $V_{u,v} = {}^{+}V_{u,v}, V_{x,y} = {}^{+}V_{x,y}, u,v,x,y \in \mathcal{A}$, and we will call $(\mathcal{A},^{-})$ an anti-structurable algebra if identity (1.14) is fulfilled for $V_{u,v} = {}^{-}V_{u,v}, V_{x,y} = {}^{-}V_{x,y}$.

Remark 1.19. If $(\mathcal{A},^{-})$ is structurable, then, in the terminology of [21], the triple system $B_{\mathcal{A}}$ is called a GJTS and by [7], $B_{\mathcal{A}}$ is a GJTS of 2nd order, i.e., satisfies the identities (1.3) and (1.4).

Definition 1.20. If $(\mathcal{A},^{-})$ is anti-structurable, then we call $B_{\mathcal{A}}$ an anti-GJTS.

Put $T_x := V_{x,1}$ for $x \in \mathcal{A}$. Then, by (1.12), $T_x = L_x + \delta R_{x-\overline{x}}$ for $x \in \mathcal{A}$, thus $T_h = L_h$, $h \in \mathcal{H}$.

Remark 1.21. (i) If $u = h \in \mathcal{H}$ and $x,y \in \mathcal{A}$, (1.14) becomes (1.15).

$$\big[L_h, V_{x,y}\big] = V_{hx,y} - V_{x,hy} \tag{1.15}$$

(ii) Suppose $^{-}$ is the identity map and hence $\mathcal{A}$ is commutative. If $(\mathcal{A},^{-})$ is δ-structurable, then $\mathcal{A}$ is a Jordan algebra, by [18]. Conversely, by [24, Section 3], any Jordan algebra satisfies (1.15) if $V_{x,y} = {}^{+}V_{x,y}$ for $x,y \in \mathcal{A}$, hence it is structurable. By (1.15) and [18], any Jordan algebra is anti-structurable if it satisfies $((hx)y)z - h((xy)z) = (x(yh))z - (xy)(hz)$ for $h,x,y,z \in \mathcal{A}$.

Clearly, the last identity is fulfilled by an associative algebra.

(iii) If $x \in \mathcal{A}$ and $T_x(1) = 0$, then $x = 0$, by [18].

Definition 1.22. For $s \in \mathcal{S}$ and $h \in \mathcal{H}$, we say that $(\mathcal{A},^{-})$ is $\mathcal{S}$ skew-alternative if $[s,x,y] = -[x,s,y]$ while $(\mathcal{A},^{-})$ is $\mathcal{H}$ skew-alternative if $[h,x,y] = -[x,h,y]$ for $x,y \in \mathcal{A}$.

Remark 1.23. If $(\mathcal{A},^-)$ is $\mathcal{S}$ skew-alternative, then by [1], $[s,x,y]=-[x,s,y]=[x,y,s],\ s\in\mathcal{S},\ x,y\in\mathcal{A}$. If $(\mathcal{A},^-)$ is $\mathcal{H}$ skew-alternative, then by (1.11), $[h,x,y]=-[x,h,y]=[x,y,h]$, $h\in\mathcal{H},\ x,y\in\mathcal{A}$.

Proposition 1.24 (see [18])**.** *If $(\mathcal{A},^-)$ is structurable, then $(\mathcal{A},^-)$ is $\mathcal{S}$ skew-alternative. If $(\mathcal{A},^-)$ is anti-structurable, then $(\mathcal{A},^-)$ is $\mathcal{H}$ skew-alternative.*

Remark 1.25. Let $(\mathcal{A},^-)$ be a δ-structurable algebra and let $\mathrm{Der}(\mathcal{A},^-)$ be the set of derivations of $\mathcal{A}$ that commute with $^-$. By Remark (iii) above $T_{\mathcal{A}}\cap\mathrm{Der}(\mathcal{A},^-)=0$ and so we may define the *structure algebra* $\mathrm{Str}(\mathcal{A},^-):=T_{\mathcal{A}}\oplus\mathrm{Der}(\mathcal{A},^-)$. This algebra plays an important role in the structure study of structurable algebras (see [1]) and may play a role in the structure study of anti-structurable algebras (theory to be presented elsewhere).

1.3 Examples

For examples of structurable algebras, we refer to [1, 2].

Definition 1.26. Let (B,U) and (B',U') be two triple systems. A linear map μ of U into U' is called a homomorphism if μ satisfies $\mu(B(x,y,z))=B'(\mu(x),\mu(y),\mu(z)),\ x,y,z\in U$. Moreover, if μ is bijective, then μ is called an isomorphism and (B,U) and (B',U') are said to be isomorphic.

Definition 1.27. Let $(A,^-)$ be a unital nonassociative algebra over Φ with involution $^-$ and let $(A^{op},^-)$ denote the opposite algebra, i.e., the algebra with multiplication defined by $x\cdot_{op}y=yx,\ x,y\in A$, where in the right-hand side of the equality the multiplication is done in A.

Remark 1.28. The algebras $(A,^-)$ and $(A^{op},^-)$ are isomorphic under the map $x\mapsto\overline{x}$.

Let L_x,R_x be defined by $L_x(y):=xy,R_x(y):=yx,\ x,y\in\mathcal{A}$. For $\delta=\pm1$, define (1.16) and (1.17).

$$^{\delta}V^{op}_{x,y}:=R_{R_x(\overline{y})}+\delta\big(L_xL_{\overline{y}}-L_yL_{\overline{x}}\big) \tag{1.16}$$

$$^{\delta}B^{op}_{\mathcal{A}}(x,y,z):=^{\delta}V^{op}_{x,y}(z)=z(\overline{y}x)+\delta\big[x(\overline{y}z)-y(\overline{x}z)\big],\quad x,y,z\in\mathcal{A} \tag{1.17}$$

Proposition 1.29. *$\mathcal{A}$ is a δ-structurable algebra if and only if $\mathcal{A}^{op}$ is a δ-structurable algebra.*

Proof. Clearly, B^{op}_A is the triple system obtained from the algebra $(\mathcal{A}^{op},^-)$, and so $B_{\mathcal{A}}$ and $B^{op}_{\mathcal{A}}$ are isomorphic under the map $x\mapsto\overline{x}$, by (1.13) and (1.17). □

Let $\mathcal{M}_{m,n}(\Phi)$ denote the vector space of $m\times n$ matrices over Φ and for $x\in\mathcal{M}_{m,n}(\Phi)$ denote by $x^\top$ the transposed matrix.

Lemma 1.30. *$(\mathcal{M}_{m,n}(\Phi),\{x,y,z\})$ is a $(-1,\delta)$-FKTS, where $\{x,y,z\}$ is defined by (1.18).*

$$\{x,y,z\}:=xy^\top z+\delta\big(zy^\top x-zx^\top y\big),\quad x,y,z\in\mathcal{M}_{m,n}(\Phi) \tag{1.18}$$

Proof. It is straightforward calculation to show that the identities (1.3) and (1.4) hold. □

Theorem 1.31. *$\mathcal{M}_{n,n}(\Phi)$ with the involution $x\mapsto x^\top$ is a δ-structurable algebra.*

Proof. It is a direct consequence of Lemma 1.30. □

Example 1.32. $(\mathcal{M}_{m,n}(\mathbb{C}),\{x,y,z\})$ is a $(-1,\delta)$-FKTS, where $\{x,y,z\}$ is defined by (1.19)

$$\{x,y,z\}:=x\overline{y}^\top z+\delta\big(z\overline{y}^\top x-z\overline{x}^\top y\big),\quad x,y,z\in\mathcal{M}_{m,n}(\mathbb{C}) \tag{1.19}$$

Indeed, it is straightforward calculation to show that the identities (1.3) and (1.4) hold. Hence $\mathcal{M}_{n,n}(\mathbf{C})$ with the involution $x \mapsto \overline{x}^\top$ is a δ-structurable algebra.

Remark 1.33. By [17], the following construction of Lie superalgebras is obtained by the standard embedding method. If $U(-1,-1) := \mathcal{M}_{2n,m}(\Phi)$ with the product (1.18), then the corresponding standard embedding Lie superalgebra is $\mathfrak{osp}(2n \mid 2m) = D(n,m)$ (as defined by [8, 9]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n,2n}(\Phi)$ is $\mathfrak{osp}(2n \mid 4n)$. Similarly, if $U(-1,-1) := \mathcal{M}_{2n+1,m}(\Phi)$ with the product (1.18), then the corresponding standard embedding Lie superalgebra is $\mathfrak{osp}(2n+1 \mid 2m) = B(n,m)$ (as defined by [8, 9]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $\mathcal{M}_{2n+1,2n+1}(\Phi)$ is $\mathfrak{osp}(2n+1 \mid 4n+2)$.

The construction of these Lie superalgebras and the correspondence with extended Dynkin diagrams is the subject of the next section. The study of the structure theory of anti-structurable algebras, the Peirce decomposition (as defined by [14, 23]), will be considered as future work. Moreover, let U be an anti-structurable algebra and associative algebra, then U is a $(-1,-1)$-FKTS. The details will be described in a future paper.

2 Anti-structurable algebras and extended Dynkin diagrams

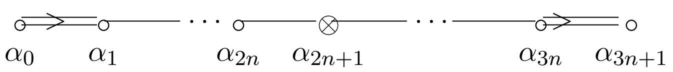
Let $U := \mathcal{M}_{l,l}(\Phi)$ with the product (1.18) and $\delta = -1$, that is, $\{x,y,z\}$ is defined by (2.1)

$$\{x,y,z\} := xy^\top z - zy^\top x + zx^\top y, \quad x,y,z \in \mathcal{M}_{l,l}(\Phi) \tag{2.1}$$

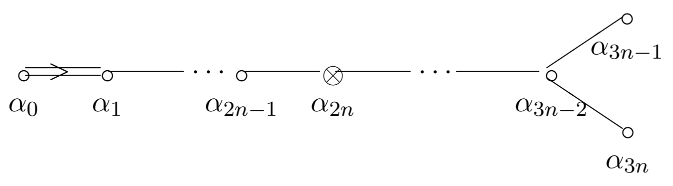
Then from the previous section this triple system is a simple unitary $(-1,-1)$-FKTS obtained from anti-structurable algebra $(U, {}^\top)$. Hence by the methods of the standard embedding associated to U we can obtain the standard embedding Lie superalgebra as follows from the following proposition: the Lie (super)algebras notations and extended Dynkin diagrams are those of [8].

Proposition 2.1. *Let $(U, {}^\top), U := \mathcal{M}_{l,l}(\Phi)$ be anti-structurable algebras and let $L(U) = \oplus_{l=-2}^{2} L_l$ be the standard embedding Lie superalgebra. Then $L(U), L_{-2} \oplus L_0 \oplus L_2, L_0$ and the corresponding extended Dynkin diagrams with $\otimes$ roots deleted are*

$$(i) \quad \begin{cases} L(U) = B(n,l) = \mathfrak{osp}(l \mid 2l) \\ L_{-2} \oplus L_0 \oplus L_2 = C_l \oplus B_n, & \textit{for } l = 2n+1 \\ L_0 = A_{l-1} \oplus B_n \oplus \lambda\Phi \end{cases}$$



$$(ii) \quad \begin{cases} L(U) = D(n,l) = \mathfrak{osp}(l \mid 2l) \\ L_{-2} \oplus L_0 \oplus L_2 = C_l \oplus D_n, & \textit{for } l = 2n \\ L_0 = A_{l-1} \oplus D_n \oplus \lambda\Phi \end{cases}$$



Proof. From $\kappa = \{x^\top z + z^\top x\}_{\text{span}} = \{A \mid A^\top = A,\ A \in \mathcal{M}_{l,l}(\Phi)\}$ and somewhat long calculations, it follows that $(U := \mathcal{M}_{l,l}(\Phi), \{\quad\})$ defined by (2.1) are simple unitary $(-1,-1)$-FKTSs. Then the standard embedding Lie superalgebras follows from [17]. Moreover, since $L_{\bar{1}} = L_{-1} \oplus L_1$ is an anti-LTS and $L_{\bar{0}} = L_{-2} \oplus L_0 \oplus L_1 = \mathrm{Der}(L_{-1} \oplus L_1)$, it is a straightforward calculation to check that $L_{\bar{0}}$ is obtained from the extended Dynkin diagram of $L(U)$ by deleting the root α_l, while L_0 is isomorphic to the corresponding Dynkin diagram (α_l deleted) $\oplus \lambda\Phi$. □

Remark 2.2. These results mean that the correspondence between anti-structurable algebras and extended Dynkin diagrams is a useful concept for the structure theory of triple systems.

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Received September 01, 2008
Revised December 09, 2008