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# On triple systems and extended Dynkin diagrams of Lie superalgebras [1]

*Noriaki KAMIYA* [a] *and Susumu OKUBO* [b]

[a] *Department of Mathematics, University of Aizu, Aizuwakamatsu, 965-8580 Fukushima, Japan*
*E-mail: kamiya@u-aizu.ac.jp*

[b] *Department of Physics, University of Rochester, Rochester, 14627 New York, U.S.A*
*E-mail: okubo@pas.rochester.edu*

**Abstract**

Our aim is to give a characterization of extended Dynkin diagrams of Lie superalgebras by means of concept of triple systems.

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## 1 Preliminaries and examples

Throughout this paper, we shall be concerned with algebras and triple systems over a field $\Phi$ that is characteristic not 2 and do not assume that our algebras and triple systems are finite dimensional, unless otherwise specified.

**Definition 1.1.** For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a vector space $U(\varepsilon, \delta)$ over $\Phi$ with the triple product $< -, -, - >$ is called a $(\varepsilon, \delta)$-*Freudenthal–Kantor triple system* if

$$[L(a,b), L(c,d)] = L(<abc>, d) + \varepsilon L(c, <bad>)$$
$$K(<abc>, d) + K(c, <abd>) + \delta K(a, K(c,d)b) = 0$$

where

$$L(a,b)c = <abc>, \quad K(a,b)c = <acb> - \delta <bca>, \quad [A,B] = AB - BA$$

*Remark* 1.1. We note that

$$S(a,b) := L(a,b) + \varepsilon L(b,a)$$
$$A(a,b) := L(a,b) - \varepsilon L(b,a)$$

are a derivation and an anti-derivation of $U(\varepsilon, \delta)$, respectively.

**Definition 1.2.** A $(\varepsilon, \delta)$-Freudenthal–Kantor triple system over $\Phi$ is said to be *balanced* if

$$\dim_\Phi \{K(x,y)\}_{\text{span}} = 1$$

**Definition 1.3.** For $\delta = \pm 1$, a triple system over $\Phi$ is said to be $\delta$-*Lie triple system* if the following are satisfied:

$$[abc] = -\delta [bac]$$
$$[abc] + [bca] + [cab] = 0$$
$$[ab[cde]] = [[abc]de] + [c[abd]e] + [cd[abe]]$$

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For the $\delta$-Lie triple systems associated with $(\varepsilon, \delta)$-Freudenthal–Kantor triple systems, we have the following.

**Proposition 1.1** ([7])**.** *Let $U(\varepsilon, \delta)$ be a $(\varepsilon, \delta)$-Freudenthal–Kantor triple system. If $P$ is a linear transformation of $U(\varepsilon, \delta)$ such that $P < xyz > = < PxPyPz >$ and $P^2 = -\varepsilon\delta\,\mathrm{Id}$, then $(U(\varepsilon, \delta), [-,-,-])$ is a Lie triple system for the case of $\delta = 1$ and an anti-Lie triple system for the case of $\delta = -1$ with respect to the product*

$$[xyz] := < xPyz > -\delta < yPxz > +\delta < xPzy > - < yPzx >$$

**Corollary 1.1.** *Let $U(\varepsilon, \delta)$ be a $(\varepsilon, \delta)$-Freudenthal–Kantor triple system. Then the vector space $T(\varepsilon, \delta) := U(\varepsilon, \delta) \oplus U(\varepsilon, \delta)$ becomes a Lie triple system for the case of $\delta = 1$ and an anti-Lie triple system for the case of $\delta = -1$ with respect to the triple product defined by*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}$$

¿From these results, it follows that the vector space

$$L(V) := \mathrm{Inn\ Der}\ T \oplus T\ \ (= L(T,T) \oplus T)$$

where $T$ is a $\delta$-Lie triple system and Inn Der $T := \{L(X,Y) | X, Y \in T\}_{\mathrm{span}}$ turns out to be a Lie algebra ($\delta = 1$) or Lie superalgebra ($\delta = -1$) by

$$[D + X, D' + X'] = [D, D'] + L(X, X') + DX' - D'X$$

**Definition 1.4.** We denote by $L(\varepsilon, \delta)$ the Lie algebras or Lie superalgebras obtained from these constructions associated with $U(\varepsilon, \delta)$ and call these algebras a *canonical standard embedding.*

**Definition 1.5.** A $(\varepsilon, \delta)$-Freudenthal–Kantor triple system $U(\varepsilon, \delta))$ is said to be *unitary* if the linear span $\mathbf{k}$ of the set $\{K(a,b) | a, b \in U(\varepsilon, \delta)\}$ contains the identity endomorphism Id.

*Remark* 1.2. We note that the balanced property is unitary.

For these standard embedding Lie algebras or superalgebras $L(\varepsilon, \delta)$, we have the following 5 grading subspaces:

$$L(\varepsilon, \delta) = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$$

where

$$U(\varepsilon, \delta) = L_{-1}, \quad T(\varepsilon, \delta) = L_{-1} \oplus L_1, \quad \mathbf{k} = \{K(a,b)\}_{\mathrm{span}} = L_{-2}$$

## 2 Lie superalgebras $D(2, 1; \alpha)$, $G(3)$ and $F(4)$

These constructions of $D(2, 1, \alpha)$, $G(3)$ and $F(4)$ are considered [8, 1]. Briefly describing, we have the following.

1. *Let $V$ be a quarternion algebra over the complex numbers. Then $V$ is a balanced $(-1, -1)$ Freudenthal–Kantor triple system with respect to certain triple product and the standard embedding Lie superalgebra $L(U)$ is $D(2, 1; \alpha)$ type's with $dim L(V) = 17$.*
2. *Let $V$ be a octonion algebra over the complex number. Then $V$ is a balanced $(-1, -1)$ Freudenthal–Kantor triple system with respect to certain triple product and the standard embedding Lie superalgebra $L(U)$ is $F(4)$ type's with $dim L(V) = 40$.*
3. *Let $V$ be a Im $\mathbb{O}$ (= the imaginary part of octonion algebra ). Then $V$ is a balanced $(-1, -1)$-Freudenthal–Kantor triple system with respect to certain triple product and the standard embedding Lie superalgebra $L(U)$ is $G(3)$ type's with $dim L(V) = 31$.*

## 3 Extended Dynkin diagrams and triple systems

In this section, we will only describe about distinguished extended Dynkin diagram of their canonical Lie superalgebras associated with $(-1,-1)$-Freudenthal–Kantor triple systems F(4) and G(3) types, because for the other cases we may deal with the explaination by means of the same methods.

(a) For F(4) type distinguished extended Dynkin diagram and usual Dynkin diagram [3] we have the following:

$$\bigcirc \equiv> \bigotimes - \bigcirc \Leftarrow \bigcirc - \bigcirc$$

$$\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$$

$$\bigotimes - \bigcirc \Leftarrow \bigcirc - \bigcirc$$

$$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$$

$U = L_{-1} = (-1,-1)$ is a balanced Freudenthal–Kantor triple system with $\dim U = 8$ (cf Sec. 2).

$$\begin{array}{llll} U \leftrightarrow \bigotimes & \bigcirc & \bigcirc & \bigcirc \\ & \bigcirc & \bigcirc & \\ & \bigcirc & & \end{array}$$

$$\leftrightarrow \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+2\alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3+\alpha_4,$$
$$\alpha_1+2\alpha_2+2\alpha_3+\alpha_4, \alpha_1+3\alpha_2+2\alpha_3+\alpha_4\}$$

$$L_{-2}(U) \leftrightarrow \{2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4\}$$

$L(U)$ is the standard embedding Lie superalgebra associated with $U$ and $\dim L(U) = 40$, $\dim L_{-2} = \dim L_2 = 1$. Then we can easily see its structure as follows:

$$L(U)/(L_{-2} \oplus L_0 \oplus L_2) \cong L_{-1} \oplus L_1 := T \qquad \text{(as anti-Lie triple system)}$$

and

$$\begin{aligned} \text{Inn Der}\, T &\cong L_{-2} \oplus L_0 \oplus L_2 = A_1 \oplus B_3 \\ &= \text{distinguished extended Dynkin diagram with omitted } \otimes \\ &= \left\{ \begin{pmatrix} L(a,b) & -K(c,d) \\ K(e,f) & -L(b,a) \end{pmatrix} \right\}_{\text{span}} \\ L_0 = \lambda I \oplus B_3 &= \left\{ \begin{pmatrix} L(a,b) & 0 \\ 0 & -L(b,a) \end{pmatrix} \right\}_{\text{span}} = \{L(a,b)\}_{\text{span}} \end{aligned}$$

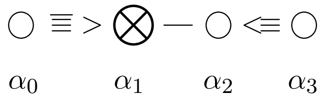
of course, $L(a,b) = S(a,b) + A(a,b)$, where $S(a,b)$ is a inner derivation of $U$, $K(a,b) = A(a,b) =<.|.>$ Id is an anti-derivation of $U$.

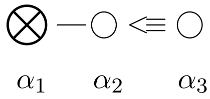
Furthermore, these imply

$$\begin{aligned} A_1 &\cong \left\{ \begin{pmatrix} 0 & \text{Id} \\ 0 & 0 \end{pmatrix} \right\}_{\text{span}} \oplus \left\{ \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \right\}_{\text{span}} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ \text{Id} & 0 \end{pmatrix} \right\}_{\text{span}} \\ &= L_{-2} \oplus \{A(a,b)\}_{\text{span}} \oplus L_2 \end{aligned}$$

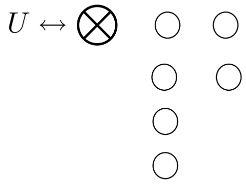
$$\text{Inn Der}\, U = \{S(a,b)\}_{\text{span}} \cong B_3 = \text{Dynkin diagram with omitted } \otimes$$

(b) For G(3) type distinguished extended Dynkin diagram and usual Dynkin diagram [3] as well as F(4) we have the following:




$U = L_{-1} = (-1,-1)$-balanced Freudenthal–Kantor triple system with $\dim U = 7$ (cf Section 2),




$$\leftrightarrow \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3, \alpha_1+3\alpha_2+\alpha_3, \alpha_1+3\alpha_2+2\alpha_3, \alpha_1+4\alpha_2+2\alpha_3\}$$

$$L_{-2}(U) \leftrightarrow \{2\alpha_1 + 4\alpha_2 + 2\alpha_3\}$$

$L(U)$ is the standard embedding Lie superalgebra associated with $U$ and $\dim L(U) = 31$, $\dim L_{-2} = \dim L_2 = 1$. Then we can easily see its structure as follows:

$$L(U)/(L_{-2} \oplus L_0 \oplus L_2) \cong L_{-1} \oplus L_1 := T \quad \text{(as anti-Lie triple system)}$$

and

$$\begin{aligned} \text{Inn Der}\, T &\cong L_{-2} \oplus L_0 \oplus L_2 = A_1 \oplus G_2 \\ &= \text{distinguished extended Dynkin diagram with omitted } \otimes \\ &= \left\{ \begin{pmatrix} L(a,b) & -K(c,d) \\ K(e,f) & -L(b,a) \end{pmatrix} \right\}_{\text{span}} \\ L_0 = \lambda I \oplus B_3 &= \left\{ \begin{pmatrix} L(a,b) & 0 \\ 0 & -L(b,a) \end{pmatrix} \right\}_{\text{span}} = \{L(a,b)\}_{\text{span}} \end{aligned}$$

Of course, $L(a,b) = S(a,b) + A(a,b)$, where $S(a,b)$ is an inner derivation of $U$, $K(a,b) = A(a,b) =<.|.>$ Id is an anti-derivation of $U$. Furthermore, these imply

$$\begin{aligned} A_1 &\cong \left\{ \begin{pmatrix} 0 & \text{Id} \\ 0 & 0 \end{pmatrix} \right\}_{\text{span}} \oplus \left\{ \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \right\}_{\text{span}} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ \text{Id} & 0 \end{pmatrix} \right\}_{\text{span}} \\ &= L_{-2} \oplus \{A(a,b)\}_{\text{span}} \oplus L_2 \\ \text{Inn Der}\, U &= \{S(a,b)\}_{\text{span}} \cong G_2 = \text{Dynkin diagram with omitted } \otimes \end{aligned}$$

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