Ashdin Publishing
Journal of Generalized Lie Theory and Applications
Vol. 4 (2010), Article ID G090401, 12 pages
doi:10.4303/jglta/G090401

Quantization of the q-analog Virasoro-like algebras

Yongsheng CHENG [a] *and Yiqian SHI* [b]

[a] *Institute of Contemporary Mathematics & College of Mathematics and Information Science, Henan University, Kaifeng 475004, China*

[b] *Department of Mathematics, University of Science and Technology of China, Hefei 230026, China*

E-mail: yscheng@ustc.edu.cn

Abstract

We use the general method of quantization by Drinfel'd twist element to quantize explicitly the Lie bialgebra structures on the q-analog Virasoro-like algebras studied in *Comm. Algebra*, **37** (2009), 1264–1274.

2000 MSC: 17B10, 17B65, 17B68

1 Introduction

The study of Lie bialgebras [1, 2] is now well established as an infinitesimalization of the notion of a quantum group or Hopf algebra. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ provided with a Lie cobracket which is related to the Lie bracket by a certain compatibility condition. According to quantum groups theory, a quantum group is essentially a formal deformation of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, the semiclassical structure associated with such a deformation is a Lie bialgebra structure on $\mathfrak{g}$. Constructing quantizations of Lie bialgebras is an important method to produce new quantum groups. Using the method twisting the coproduct by a Drinfel'd twist element but keeping the product unchanged, Grunspan [3] presented the quantization of a class of infinite dimensional Lie algebras containing Virasoro algebras studied in [4] (see also [5, 6]). Using the same technique, Hu and Wang [7] quantized some Lie algebras presented in [8]. In a recent paper [9], the Lie bialgebra structures of q-analog Virasoro-like algebras $\mathfrak{L}$ with the basis $\{L_\alpha, d_1, d_2 \mid \alpha \in \mathbb{Z}^2\backslash\{(0,0)\}\}$ and brackets

$$\big[L_\alpha, L_\beta\big] = \big(q^{\alpha_2\beta_1} - q^{\alpha_1\beta_2}\big)L_{\alpha+\beta}, \quad \big[d_i, L_\alpha\big] = \alpha_i L_\alpha,\ i = 1, 2, \tag{1.1}$$

for $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2\backslash\{(0,0)\}$, were considered, where $0 \neq q \in \mathbb{C}$ is a fixed non-root of unity. Here we treat $L_{0,0}$ as zero. Obviously, the Lie algebra $\mathfrak{L}$ is $\mathbb{Z}^2$-graded (however its structural constant $q^{\alpha_2\beta_1} - q^{\alpha_1\beta_2}$ is not linearly dependent on the gradings α, β; in this case, the Lie algebra $\mathfrak{L}$ is called *non-linear*). This Lie algebra is closely related to the Virasoro and Virasoro-like algebras and the Lie algebras of Cartan type S and H (cf. [15, 16]), which is probably why this type of Lie algebras has attracted some attentions in the literature (cf. [10, 11, 12, 13, 14, 17, 18]).

In this paper, we will use the techniques developed in [3, 7] to construct the quantization of this type of bialgebra. However, since in our case the Lie algebra is non-linear, some of our arguments may render rather technical.

We fix a field $\mathbb{F}$ of characteristic zero. Let $\mathcal{A}$ be a unitary $\mathcal{R}$-algebra ($\mathcal{R}$ is a ring). For $z \in \mathcal{A}$, $n \in \mathbb{Z}$, we set

$$z^{\langle n\rangle} = z(z+1)\cdots(z+n-1), \quad z^{[n]} = z(z-1)\cdots(z-n+1)$$

and set $z^{\langle 0\rangle} = 1$ and $z^{[0]} = 1$. If $a \in \mathcal{R}$ is any scalar, set $z_a^{\langle n\rangle} = (z+a)^{\langle n\rangle}$ and $z_a^{[n]} = (z+a)^{[n]}$, that is

$$z_a^{\langle n\rangle} = (z+a)(z+a+1)\cdots(z+a+n-1), \tag{1.2}$$

$$z_a^{[n]} = (z+a)(z+a-1)\cdots(z+a-n+1). \tag{1.3}$$

Obviously $z^{\langle n\rangle} = z_0^{\langle n\rangle}$, $z^{[n]} = z_0^{[n]}$.

The following lemma can be found in [3].

Lemma 1.1. *Let z be any element of a unitary $\mathbb{F}$-algebras $\mathcal{A}$. For $a, d \in \mathbb{F}$, and $m, n, r \in \mathbb{Z}$, one has*

$$z_a^{\langle m+n\rangle} = z_a^{\langle m\rangle} z_{a+m}^{\langle n\rangle}, \quad z_a^{[m+n]} = z_a^{[m]} z_{a-m}^{[n]}, \quad z_a^{[m]} = z_{a-m+1}^{\langle m\rangle}, \tag{1.4}$$

$$\begin{aligned} &\sum_{m+n=r} \frac{(-1)^n}{m!n!} z_a^{[m]} z_d^{\langle n\rangle} = \binom{a-d}{r}, \\ &\sum_{m+n=r} \frac{(-1)^n}{m!n!} z_a^{[m]} z_{d-m}^{[n]} = \binom{a-d+r-1}{r}, \end{aligned} \tag{1.5}$$

where in general $\binom{a}{b}$ is the binomial coefficient.

Denote by $(U(\mathfrak{L}), \mu, \tau, \Delta_0, S_0, \epsilon_0)$ the natural Hopf algebra structure on $U(\mathfrak{L})$ (the universal enveloping algebra of the Lie algebra $\mathfrak{L}$), that is, the coproduct Δ_0, the antipode S_0 and the counit ϵ_0 are respectively defined by

$$\begin{aligned} &\Delta_0\big(L_\beta\big) = L_\beta \otimes 1 + 1 \otimes L_\beta, \quad \Delta_0\big(d_i\big) = d_i \otimes 1 + 1 \otimes d_i, \\ &S_0\big(L_\beta\big) = -L_\beta, \quad S_0\big(d_i\big) = -d_i, \\ &\epsilon_0\big(L_\beta\big) = 0, \quad \epsilon_0\big(d_i\big) = 0 \quad \text{for } \beta \in \mathbb{Z}^2 \setminus \{(0,0)\},\ i = 1, 2. \end{aligned}$$

The following definition and well-known result can be found in [2].

Definition 1.2. Let $(\mathcal{H}, \mu, \tau, \Delta_0, S_0, \epsilon_0)$ be a Hopf algebra over a commutative ring. An element $\mathscr{F} \in \mathcal{H} \otimes \mathcal{H}$ is called Drinfel'd twist element, if it is invertible such that

$$(\mathscr{F} \otimes 1)\big(\Delta_0 \otimes Id\big)(\mathscr{F}) = (1 \otimes \mathscr{F})\big(Id \otimes \Delta_0\big)(\mathscr{F}), \tag{1.6}$$

$$(\epsilon_0 \otimes Id)(\mathscr{F}) = 1 \otimes 1 = \big(Id \otimes \epsilon_0\big)(\mathscr{F}). \tag{1.7}$$

Lemma 1.3. *Let $(\mathcal{H}, \mu, \tau, \Delta_0, S_0, \epsilon_0)$ be a Hopf algebra over a commutative ring, and let $\mathscr{F}$ be a Drinfel'd twist element of $\mathcal{H} \otimes \mathcal{H}$, then*

(1) $\mathscr{U} = \mu(Id \otimes S_0)(\mathscr{F})$ is an invertible element of $\mathcal{H}$ with $\mathscr{U}^{-1} = \mu(S_0 \otimes Id)(\mathscr{F}^{-1})$;

(2) the algebra $(\mathcal{H}, \mu, \tau, \Delta, S, \epsilon)$ is a new Hopf algebra if we keep the counit undeformed (i.e., $\epsilon = \epsilon_0$) and define $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, $S : \mathcal{H} \to \mathcal{H}$ by

$$\Delta(h) = \mathscr{F}\Delta_0(h)\mathscr{F}^{-1}, \quad S(h) = uS_0(h)u^{-1}.$$

Let $(U(\mathfrak{g}), \mu, \tau, \Delta_0, S_0, \epsilon_0)$ be the natural Hopf algebra structure, where $\mathfrak{g}$ is a triangular Lie bialgebra, and denote by $U(\mathfrak{g})[[t]]$ an associative $\mathbb{F}$-algebra of formal power series with coefficients in $U(\mathfrak{g})$. Naturally, $U(\mathfrak{g})[[t]]$ is equipped with an induced Hopf algebra structure arising from that on $U(\mathfrak{g})$.

Definition 1.4. For a triangular Lie bialgebra $\mathfrak{g}$ over $\mathbb{F}$, the Hopf algebra $(U(\mathfrak{g})[[t]], \mu, \tau, \Delta_r, S_r, \epsilon_0)$ is called a quantization of $(U(\mathfrak{g}), \mu, \tau, \Delta_0, S_0, \epsilon_0)$ by a Drinfel'd twist element $\mathscr{F}$, if $U(\mathfrak{g})[[t]]/tU(\mathfrak{g})[[t]] \cong U(\mathfrak{g})$ and $\mathscr{F}$ is determined by its r-matrix r.

We will fix the following notations, for $x_1, x_2 \in \mathbb{Z}$,

$$\begin{aligned} &T = x_1 d_1 + x_2 d_2 \in \operatorname{span}\{d_1, d_2\}, \\ &E = L_\alpha \text{ for } \alpha \in \mathbb{Z}^2 \setminus (0,0) \quad \text{satisfying } [T, E] = E. \end{aligned} \tag{1.8}$$

The following result is obtained in [9].

Lemma 1.5. *There is a triangular Lie bialgebra structure on the Lie algebras $\mathfrak{L}$ given by the r-matrix $T \otimes E - E \otimes T$, where T and E are defined in* (1.8).

The main result of this paper is the following theorem.

Theorem 1.6. *Let $\mathfrak{L}$ be the q-analog Virasoro-like algebras with $[T, E] = E$ (cf. (1.8)), then there exists a noncommutative and noncocommutative Hopf algebra structure $(U(\mathfrak{L})[[t]], \mu, \tau, \Delta, S, \epsilon)$ on $U(\mathfrak{L})[[t]]$, such that $U(\mathfrak{L})[[t]]/tU(\mathfrak{L})[[t]] = U(\mathfrak{L})$, which preserves the product and the counit of $U(\mathfrak{L})[[t]]$, but the coproduct and antipode are defined by*

$$\Delta\big(L_\beta\big) = L_\beta \otimes (1 - Et)^c + \sum_{k=0}^{\infty} (-1)^k a_k T^{\langle k \rangle} \otimes (1 - Et)^{-k} L_{\beta + k\alpha} t^k, \tag{1.9}$$

$$\Delta\big(d_i\big) = d_i \otimes 1 + 1 \otimes d_i + \alpha_i T \otimes (1 - Et)^{-1} Et, \tag{1.10}$$

$$S\big(L_\beta\big) = -(1 - Et)^{-c} \sum_{k=0}^{\infty} a_k L_{\beta + k\alpha} T_1^{\langle k \rangle} t^k, \tag{1.11}$$

$$S\big(d_i\big) = \alpha_i T (1 - Et)^{-1} \big(Et - E^2 t^2\big) - d_i, \tag{1.12}$$

where

$$c = x_1 \beta_1 + x_2 \beta_2, \quad a_k = \frac{1}{k!} \prod_{p=1}^{k} \big(q^{\alpha_2(\beta_1 + (p-1)\alpha_1)} - q^{\alpha_1(\beta_2 + (p-1)\alpha_2)}\big), \quad c_0 = 1, \quad i = 1, 2.$$

In fact, we can introduce the operator $\mathscr{D}_{(n)}$ $(n \in \mathbb{N})$ on $U(\mathfrak{L})$ defined by $\mathscr{D}_{(n)} := \frac{1}{n!}(\operatorname{ad} E)^n$; it is easy to check that

$$\mathscr{D}_{(n)}\big(L_\beta\big) = a_n L_{\beta + n\alpha}. \tag{1.13}$$

Thus, (1.9) and (1.11) in Theorem 1.6 can be rewritten as

$$\Delta\big(L_\beta\big) = L_\beta \otimes (1 - Et)^c + \sum_{p=0}^{\infty} (-1)^p T^{\langle p \rangle} \otimes (1 - Et)^{-p} \mathscr{D}_{(p)}\big(L_\beta\big) t^p, \tag{1.14}$$

$$S\big(L_\beta\big) = -(1 - Et)^{-c} \sum_{p=0}^{\infty} \mathscr{D}_{(p)}\big(L_\beta\big) T_1^{\langle p \rangle} t^p. \tag{1.15}$$

2 Proof of the main results

From above, in order to quantize the Lie bialgebra structures on q-analog Virasoro-like algebras, the key is to construct the Drinfel'd twisting, thus we have to do some necessary computation.

Lemma 2.1. *Let $\mathfrak{L}$ be the q-analog Virasoro-like algebras. The following equations hold in $U(\mathfrak{L})$:*

$$L_\beta T_a^{[m]} = T_{a-c}^{[m]} L_\beta, \quad L_\beta T_a^{\langle m\rangle} = T_{a-c}^{\langle m\rangle} L_\beta, \tag{2.1}$$

$$E^n T_a^{[m]} = T_{a-n}^{[m]} E^n, \quad E^n T_a^{\langle m\rangle} = T_{a-n}^{\langle m\rangle} E^n, \tag{2.2}$$

$$d_n^k T_a^{[m]} = T_a^{[m]} d_n^k, \quad d_n^k T_a^{\langle m\rangle} = T_a^{\langle m\rangle} d_n^k, \tag{2.3}$$

$$L_\beta L_\gamma^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \prod_{p=1}^{i} \big(q^{\gamma_2(\beta_1+(p-1)\gamma_1)} - q^{\gamma_1(\beta_2+(p-1)\gamma_2)}\big) L_\gamma^{m-i} L_{\beta+i\gamma}, \tag{2.4}$$

$$d_n L_\gamma^m = m\gamma_n L_\gamma^m + L_\gamma^m d_n, \tag{2.5}$$

where $T = x_1 d_1 + x_2 d_2 \in \mathrm{span}\{d_1, d_2\}$, $E = L_\alpha$ satisfying $[T, E] = E$ (cf. (1.8)), $\beta, \gamma \in \mathbb{Z}^2 \setminus \{(0,0)\}$, $c = x_1\beta_1 + x_2\beta_2$, $a \in \mathbb{C}$ and $n = 1, 2$.

Proof. Since $[T, L_\beta] = cL_\beta$, we have $L_\beta T = (T - c)L_\beta$. It is easy to see that (2.1) is true for $m = 1$. We can suppose that the first equation of (2.1) is true for m, then for $m+1$, we have

$$\begin{aligned} L_\beta T_a^{[m+1]} &= L_\beta T_a^{[m]}(T + a - m) = T_{a-c}^{[m]} L_\beta (T + a - m) \\ &= T_{a-c}^{[m]}(T + a - c - m) L_\beta = T_{a-c}^{[m+1]} L_\beta. \end{aligned}$$

Thus we get (2.1) by induction on m. The second equation in (2.1), (2.2) and (2.3) can be verified in a similar way. Since

$$\big(\mathrm{ad}\, L_\gamma\big)^i L_\beta = \prod_{p=1}^{i} \big(q^{\gamma_2(\beta_1+(p-1)\gamma_1)} - q^{\gamma_1(\beta_2+(p-1)\gamma_2)}\big) L_{\beta+i\gamma}, \tag{2.6}$$

for any $L_\beta, L_\gamma \in \mathfrak{L}$, then for (2.4), we have

$$\begin{aligned} L_\beta L_\gamma^m &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} L_\gamma^{m-i} \big(\mathrm{ad}\, L_\gamma\big)^i \big(L_\beta\big) \\ &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \prod_{p=1}^{i} \big(q^{\gamma_2(\beta_1+(p-1)\gamma_1)} - q^{\gamma_1(\beta_2+(p-1)\gamma_2)}\big) L_\gamma^{m-i} L_{\beta+i\gamma}. \end{aligned}$$

Similarly, we can obtain (2.5). □

For $a \in \mathbb{F}$, we set

$$\begin{aligned} &\mathscr{F}_a := \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_a^{[i]} \otimes E^i t^i, \quad F_a := \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{\langle i\rangle} \otimes E^i t^i, \\ &\mathscr{U}_a := \mu \cdot \big(S_0 \otimes Id\big)\big(F_a\big), \quad \mathscr{V}_a := \mu \cdot \big(Id \otimes S_0\big)\big(\mathcal{F}_a\big), \end{aligned}$$

where t denotes a formal variable. Denote $\mathscr{F} = \mathscr{F}_0$, $F = F_0$, $\mathscr{U} = \mathscr{U}_0$, $\mathscr{V} = \mathscr{V}_0$. Since $S_0(T_a^{\langle i\rangle}) = (-1)^i T_{-a}^{[i]}$, $S_0(E^i) = (-1)^i E^i$, we have

$$\mathscr{U}_a = \mu\big(S_0 \otimes Id\big)\left(\sum_{i=0}^{\infty} \frac{1}{i!} T_a^{\langle i\rangle} \otimes E^i t^i\right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a}^{[i]} E^i t^i,$$

$$\mathscr{V}_a = \mu\big(Id \otimes S_0\big)\left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_a^{[i]} \otimes E^i t^i\right) = \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} E^i t^i.$$

Lemma 2.2. *For $a, d \in \mathbb{C}$, one has*

$$\mathscr{F}_a F_d = 1 \otimes (1 - Et)^{(a-d)}, \quad \mathscr{V}_a \mathscr{U}_d = (1 - Et)^{-(a+d)}. \tag{2.7}$$

Therefore the elements $\mathscr{F}_a$, F_a, $\mathscr{U}_a$, $\mathscr{V}_a$ are invertible elements with $\mathscr{F}_a^{-1} = F_a$, $\mathscr{U}_a^{-1} = \mathscr{V}_{-a}$.

Proof. Using the formula (1.5), we have

$$\begin{aligned}
\mathscr{F}_a F_d &= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{i+j=m} \frac{(-1)^j}{i!j!} T_a^{[i]} T_d^{\langle j\rangle}\right) \otimes E^m t^m \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{a-d}{m} \otimes E^m t^m \\
&= 1 \otimes (1 - Et)^{a-d}.
\end{aligned}$$

For the second equation, using (2.2) and (1.5), we have

$$\begin{aligned}
\mathscr{V}_a \mathscr{U}_d &= \sum_{m=0}^{\infty} \left(\sum_{i+j=m} \frac{(-1)^j}{i!j!} T_a^{[i]} T_{-d-i}^{[j]}\right) E^{i+j} t^{i+j} \\
&= \sum_{m=0}^{\infty} \binom{a+d+m-1}{m} E^m t^m \\
&= (1 - Et)^{-(a+d)}.
\end{aligned}$$

□

Lemma 2.3. *For any positive integer m and any $a \in \mathbb{F}$, one has*

$$\Delta_0\big(T^{[m]}\big) = \sum_{i=0}^{m} \binom{m}{i} T_{-a}^{[i]} \otimes T_a^{[m-i]}. \tag{2.8}$$

In particular, one has

$$\Delta_0\big(T^{[m]}\big) = \sum_{i=0}^{m} \binom{m}{i} T^{[i]} \otimes T^{[m-i]}.$$

Proof. In order to get the result, we want to use induction. Since $\Delta_0(T) = T\otimes 1 + 1\otimes T$, it is easy to see that the result is true for $m = 1$; suppose that it is true for m, then it is enough to consider the condition for $m+1$,

$$\begin{aligned}
\Delta_0\big(T^{[m+1]}\big) &= \Delta_0\big(T^{[m]}\big)\Delta_0(T-m)\\
&= \left(\sum_{i=0}^{m}\binom{m}{i} T_{-a}^{[i]}\otimes T_a^{[m-i]}\right)\\
&\quad\times\big((T-a-m)\otimes 1 + 1\otimes (T+a-m) + m(1\otimes 1)\big)\\
&= 1\otimes T_a^{[m+1]} + T_{-a}^{[m+1]}\otimes 1 + m\left(\sum_{i=1}^{m-1}\binom{m}{i} T_{-a}^{[i]}\otimes T_a^{[m-i]}\right)\\
&\quad + (T-a)\otimes T_a^{[m]} + T_{-a}^{[m]}\otimes (T+a) + \sum_{i=1}^{m-1}\binom{m}{i} T_{-a}^{[i+1]}\otimes T_a^{[m-i]}\\
&\quad + \sum_{i=1}^{m-1}(i-m)\binom{m}{i} T_{-a}^{[i]}\otimes T_a^{[m-i]} + \sum_{i=1}^{m-1}\binom{m}{i} T_{-a}^{[i]}\otimes T_a^{[m-i+1]}\\
&\quad + \sum_{i=1}^{m-1}(-i)\binom{m}{i} T_{-a}^{[i]}\otimes T_a^{[m-i]}\\
&= 1\otimes T_a^{[m+1]} + T_{-a}^{[m+1]}\otimes 1 + \sum_{i=1}^{m}\left(\binom{m}{i-1}+\binom{m}{i}\right) T_{-a}^{[i]}\otimes T_a^{[m+1-i]}\\
&= \sum_{i=0}^{m+1}\binom{m+1}{i} T_{-a}^{[i]}\otimes T_a^{[m+1-i]}.
\end{aligned}$$

Therefore, the result is proved by induction. ☐

Proposition 2.4. *$\mathscr{F} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}T^{[i]}\otimes E^i t^i$ is a Drinfel'd twist element of $(U(\mathfrak{L})[[t]], \mu, \tau, \Delta_0, S_0, \epsilon_0)$, that is $\mathscr{F}$ satisfies (1.6) and (1.7).*

Proof. The proof of (1.7) is easy, we just need to check (1.6). Since

$$\begin{aligned}
(\mathscr{F}\otimes 1)\big(\Delta_0\otimes Id\big)(\mathscr{F}) &= \left(\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}T^{[i]}\otimes E^i t^i\otimes 1\right)\\
&\quad\cdot\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\sum_{k=0}^{j}\binom{j}{k} T_{-i}^{[k]}\otimes T_i^{[j-k]}\otimes E^j t^j\right)\\
&= \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!j!}\sum_{k=0}^{j}\binom{j}{k} T^{[i]}T_{-i}^{[k]}\otimes E^i T_i^{[j-k]}\otimes E^j t^{i+j}\\
&= \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}}{i!j!}\sum_{k=0}^{j}\binom{j}{k} T^{[i+k]}\otimes T^{[j-k]}E^i\otimes E^j t^{i+j},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}(1\otimes\mathscr{F})\big(Id\otimes\Delta_0\big)(\mathscr{F}) &= \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}1\otimes T^{[r]}\otimes E^r t^r\right)\\ &\quad\cdot\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}T^{[s]}\otimes\sum_{q=0}^{s}\binom{s}{q}E^q\otimes E^{s-q}t^s\right)\\ &=\sum_{r,s=0}^{\infty}\frac{(-1)^{r+s}}{r!s!}\sum_{q=0}^{s}\binom{s}{q}T^{[s]}\otimes T^{[r]}E^q\otimes E^{r+s-q}t^{r+s},\end{aligned}$$

thus, to verify (1.6), it suffices to show for a fixed m that

$$\begin{aligned}&\sum_{i+j=m}\frac{1}{i!j!}\sum_{k=0}^{j}\binom{j}{k}T^{[i+k]}\otimes T^{[j-k]}E^i\otimes E^j\\ &\quad=\sum_{r+s=m}\frac{1}{r!s!}\sum_{q=0}^{s}\binom{s}{q}T^{[s]}\otimes T^{[r]}E^q\otimes E^{r+s-q}.\end{aligned}$$

Now, fix r, s, q such that $r+s=m$, $0\le q\le s$, set $i=q$, $i+k=s$, then we have $j=m-q$, $j-k=r$. We see that the coefficients of $T^{[s]}\otimes T^{[r]}E^q\otimes E^{m-q}$ in both sides are equal. So the result follows. □

Lemma 2.5. *One has for any $a\in\mathbb{F}$ and $L_\beta\in\mathfrak{L}$*

$$\big(L_\beta\otimes 1\big)F_a = F_{a-c}\big(L_\beta\otimes 1\big), \tag{2.9}$$

$$\big(1\otimes L_\beta\big)F_a = \sum_{l=0}^{\infty}(-1)^l a_l F_{a+l}\big(T_a^{\langle l\rangle}\otimes L_{\beta+l\alpha}t^l\big), \tag{2.10}$$

$$L_\beta\mathscr{U}_a = \mathscr{U}_{a+c}\sum_{l=0}^{\infty}a_l L_{\beta+l\alpha}T_{1-a}^{\langle l\rangle}t^l, \tag{2.11}$$

$$\big(d_i\otimes 1\big)F_a = F_a\big(d_i\otimes 1\big), \tag{2.12}$$

$$\big(1\otimes d_i\big)F_a = F_{a+1}\big(T_a^{\langle 1\rangle}\otimes\alpha_i Et\big)+F_a\big(1\otimes d_i\big), \tag{2.13}$$

$$d_i\mathscr{U}_a = -\alpha_i T_{-a}^{[1]}\mathscr{U}_{a+1}Et+\mathscr{U}_a d_i, \tag{2.14}$$

$$E\mathscr{U}_a = \mathscr{U}_{a+1}E, \tag{2.15}$$

$$\mathscr{V}_a T_{-a}^{[1]} = T_{-a}^{[1]}\mathscr{V}_a - T_a^{[1]}\mathscr{V}_{a-1}Et, \tag{2.16}$$

where

$$a_l=\frac{1}{l!}\prod_{p=1}^{k}\big(q^{\alpha_2(\beta_1+(p-1)\alpha_1)}-q^{\alpha_1(\beta_2+(p-1)\alpha_2)}\big),\quad c=x_1\beta_1+x_2\beta_2,\ i=1,2.$$

Proof. By the second equation of (2.1) we have

$$\big(L_\beta\otimes 1\big)F_a=\sum_{i=0}^{\infty}\frac{1}{i!}L_\beta T_a^{\langle i\rangle}\otimes E^i t^i$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} \frac{1}{i!} T_{a-c}^{\langle i\rangle} L_\beta \otimes E^i t^i \\
&= F_{a-c}\big(L_\beta \otimes 1\big);
\end{aligned}
$$

this prove (2.12). For (2.10), using (2.4), we have

$$
\begin{aligned}
\big(1 \otimes L_\beta\big) F_a &= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{\langle i\rangle} \otimes L_\beta E^i t^i \\
&= \sum_{i=0}^{\infty} \sum_{l=0}^{i} (-1)^l \frac{1}{(i-l)!} a_l T_a^{\langle i\rangle} \otimes E^{i-l} L_{\beta+l\alpha} t^i \\
&= \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{1}{i!} a_l T_a^{\langle i+1\rangle} \otimes E^i L_{\beta+l\alpha} t^{i+l} \\
&= \sum_{l=0}^{\infty} (-1)^l a_l \sum_{i=0}^{\infty} \frac{1}{i!} T_{a+l}^{\langle i\rangle} \otimes E^i t^i T_a^{\langle l\rangle} \otimes L_{\beta+l\alpha} t^l \\
&= \sum_{l=0}^{\infty} (-1)^l a_l F_{a+l}\big(T_a^{\langle l\rangle} \otimes L_{\beta+l\alpha} t^l\big);
\end{aligned}
$$

this proves (2.10). The following two equations give the proofs of (2.11) and (2.12):

$$
\begin{aligned}
L_\beta \mathscr{U}_a &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a-c}^{[r]} L_\beta E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a-c}^{[r]} \sum_{l=0}^{r} (-1)^l \frac{r!}{(r-l)!} a_l E^{r-l} L_{\beta+l\alpha} t^r \\
&= \sum_{r,l=0}^{\infty} \frac{(-1)^r}{r!} a_l T_{-a-c}^{[r+l]} E^r L_{\beta+l\alpha} t^{r+l} \\
&= \sum_{r,l=0}^{\infty} \frac{(-1)^r}{r!} a_l T_{-a-c}^{[r]} T_{-a-c-r}^{[l]} E^r L_{\beta+l\alpha} t^{r+l} \\
&= \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \left(\frac{(-1)^r}{r!} a_l T_{-a-c}^{[r]} E^r t^r \right) T_{-a-c}^{[l]} L_{\beta+l\alpha} t^l \\
&= \mathscr{U}_{a+c} \sum_{l=0}^{\infty} a_l T_{-a-c}^{[l]} L_{\beta+l\alpha} t^l \\
&= \mathscr{U}_{a+c} \sum_{l=0}^{\infty} a_l L_{\beta+l\alpha} T_{1-a}^{\langle l\rangle} t^l,
\end{aligned}
$$

$$
\begin{aligned}
\big(d_i \otimes 1\big) F_a &= \big(d_i \otimes 1\big) \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} d_i T_a^{\langle r\rangle} \otimes E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} d_i \otimes E^r t^r
\end{aligned}
$$

$$= \left(\sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes E^r t^r\right) \big(d_i \otimes 1\big)$$
$$= F_a\big(d_i \otimes 1\big).$$

Using (1.4) and (2.5), we have

$$\begin{aligned}
\big(1 \otimes d_i\big) F_a &= \big(1 \otimes d_i\big) \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes d_i E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes \big(r\alpha_i E^r + E^r d_i\big) t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{(r-1)!} T_a^{\langle r\rangle} \otimes \alpha_i E^r t^r + \sum_{r=0}^{\infty} \frac{1}{r!} T_a^{\langle r\rangle} \otimes E^r d_i t^r \\
&= \sum_{r=0}^{\infty} \frac{1}{(r-1)!} T_a^{\langle 1\rangle} T_{a+1}^{\langle r-1\rangle} \otimes \alpha_i E^r t^r + F_a\big(1 \otimes d_i\big) \\
&= F_{a+1}\big(T_a^{\langle 1\rangle} \otimes \alpha_i E t\big) + F_a\big(1 \otimes d_i\big),
\end{aligned}$$

which gives (2.12). The equations (2.14) and (2.15) follow from the following computations:

$$\begin{aligned}
d_i \mathscr{U}_a &= d_i \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} d_i E^r t^r \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} \big(r\alpha_i E^r + E^r d_i\big) t^r \\
&= \sum_{r=0}^{\infty} \alpha_i T_{-a}^{[1]} \frac{(-1)^r}{(r-1)!} T_{-a-1}^{[r-1]} E^r t^r + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} T_{-a}^{[r]} E^r t^r d_i \\
&= -\alpha_i T_{-a}^{[1]} \mathscr{U}_{a+1} E t + \mathscr{U}_a d_i, \\
E \mathscr{U}_a &= E \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a}^{[i]} E^i t^i \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} T_{-a-1}^{[i]} E^{i+1} t^i = \mathscr{U}_{a+1} E.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathscr{V}_a T_{-a}^{[1]} &= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} E^i t^i T_{-a}^{[1]} \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} T_a^{[i]} (T - a - i) E^i t^i
\end{aligned}$$

$$
\begin{aligned}
&= T_{-a}^{[1]}\mathscr{V}_a - \sum_{i=0}^{\infty}\frac{1}{(i-1)!}(T+a)T_{a-1}^{[i-1]}E^it^i\\
&= T_{-a}^{[1]}\mathscr{V}_a - T_a^{[1]}\mathscr{V}_{a-1}Et,
\end{aligned}
$$

which proves the last equation of the lemma. □

Now we can prove our main theorem in this paper.

Proof of Theorem 1.6. For arbitrary elements, $L_\beta \in \mathfrak{L}$, $i = 1, 2$. First, using (2.7), (2.12) and (2.10), we have

$$
\begin{aligned}
\Delta\big(L_\beta\big) &= \mathscr{F}\Delta_0\big(L_\beta\big)\mathscr{F}^{-1}\\
&= \mathscr{F}\big(L_\beta\otimes 1\big)\mathscr{F}^{-1} + \mathscr{F}\big(1\otimes L_\beta\big)\mathscr{F}^{-1}\\
&= \mathscr{F}F_{-c}\big(L_\beta\otimes 1\big) + \mathscr{F}\sum_{l=0}^{\infty}(-1)^l a_l F_l\big(T^{\langle l\rangle}\otimes L_{\beta+l\alpha}t^l\big)\\
&= \big(1\otimes(1-Et)^c\big)\big(L_\beta\otimes 1\big)\\
&\quad + \sum_{l=0}^{\infty}(-1)^l a_l\big(1\otimes(1-Et)^{-l}\big)\otimes\big(T^{\langle l\rangle}\otimes L_{\beta+l\alpha}t^l\big)\\
&= L_\beta\otimes(1-Et)^c + \sum_{l=0}^{\infty}(-1)^l a_l T^{\langle l\rangle}\otimes(1-Et)^{-l}L_{\beta+l\alpha}t^l.
\end{aligned}
$$

Using (2.7), (2.12) and (2.13), we have

$$
\begin{aligned}
\Delta\big(d_i\big) &= \mathscr{F}\Delta\big(d_i\big)\mathscr{F}^{-1}\\
&= \mathscr{F}\big(d_i\otimes 1 + 1\otimes d_i\big)F\\
&= \mathscr{F}\big(d_i\otimes 1\big)F + \mathscr{F}\big(1\otimes d_i\big)F\\
&= \mathscr{F}F\big(d_i\otimes 1\big) + \mathscr{F}\big(F_1\big(T^{\langle 1\rangle}\otimes\alpha_i Et\big) + F\big(1\otimes d_i\big)\big)\\
&= d_i\otimes 1 + 1\otimes d_i + 1\otimes(1-Et)^{-1}\big(T^{\langle 1\rangle}\otimes\alpha_i Et\big)\\
&= d_i\otimes 1 + 1\otimes d_i + \alpha_i T^{\langle 1\rangle}\otimes(1-Et)^{-1}Et.
\end{aligned}
$$

Using (2.7) and (2.11), we have

$$
\begin{aligned}
S\big(L_\beta\big) &= \mathscr{U}^{-1}S_0\big(L_\beta\big)\mathscr{U}\\
&= -\mathscr{V}L_\beta\mathscr{U}\\
&= -\mathscr{V}\mathscr{U}_b\left(\sum_{l=0}^{\infty}a_l L_{\beta+l\alpha}T_1^{\langle l\rangle}t^l\right)\\
&= -(1-Et)^{-b}\left(\sum_{l=0}^{\infty}a_l L_{\beta+l\alpha}T_1^{\langle l\rangle}t^l\right).
\end{aligned}
$$

Using (2.7), (2.14), (2.15) and (2.16), we have

$$
\begin{aligned}
S\big(d_i\big) &= \mathscr{U}^{-1}S_0\big(d_i\big)\mathscr{U}\\
&= -\mathscr{V}d_i\mathscr{U}\\
&= -\mathscr{V}\big(-\alpha_i T^{[1]}\mathscr{U}_1 Et + \mathscr{U}d_i\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_i\big(T\mathscr{V} - T\mathscr{V}Et\big)\mathscr{U}_1 Et - d_i \\
&= \alpha_i T\mathscr{V}\mathscr{U}_1 Et - \alpha_i T\mathscr{V}\mathscr{U}_2 E^2t^2 - d_i \\
&= \alpha_i T(1-Et)^{-1}Et - \alpha_i T(1-Et)^{-1}E^2t^2 - d_i \\
&= \alpha_i T(1-Et)^{-1}\big(Et - E^2t^2\big) - d_i.
\end{aligned}
$$

This completes the proof of the theorem. □

Acknowledgments

This work is supported by the National Science Foundation of China (No. 10825101), the Postdoctoral Science Foundation of China (No. 20090450810), the Natural Science Foundation of Henan Provincial Education Department of China (No. 2010B110003) and the Natural Science Foundation of Henan University of China (No. 2009YBZR025).

References

[1] V. Drinfel'd, Constant quasiclassical solutions of the Yang-Baxter quantum equation, *Dokl. Akad. Nauk SSSR*, **273** (1983), 531–535.

[2] V. Drinfel'd, Quantum groups, *Proceedings of the International Congress of Mathematicians* (Berkeley, CA, 1986), 798–820, American Mathematical Society, Providence, RI, USA, 1987.

[3] C. Grunspan, Quantizations of the Witt algebra and of simple Lie algebras in characteristic p, *J. Algebra*, **280** (2004), 145–161.

[4] W. Michaelis, A Class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, *Adv. Math.*, **107** (1994), 365–392.

[5] E. J. Taft, Witt and Virasoro algebras as Lie bialgebras, *J. Pure Appl. Algebra*, **87** (1993), 301–312.

[6] S. H. Ng and Earl J. Taft, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, *J. Pure Appl. Algebra*, **151** (2000), 67–88.

[7] N. Hu and X. Wang, Quantizations of generalized-Witt algebra and of Jacobson-Witt algebra in modular case, *J. Algebra*, **312** (2007), 902–929.

[8] G. Song and Y. Su, Lie bialgebras of generalized Witt type, *Sci. China Ser. A*, **49** (2006), 533–544.

[9] Y. Cheng and Y. Shi, Lie bialgebra structures on the q-analog Virasoro-like algebras, *Comm. Algebra*, **37** (2009), 1264–1274.

[10] W. Lin and S. Tan, Harish-Chandra modules for the q-analog Virasoro-like algebra, *J. Algebra*, **297** (2006), 254–272.

[11] R. Shen and Y. Su, Classification of irreducible weight modules with a finite-dimensional weight space over twisted Heisenberg-Virasoro algebra, *Acta Math. Sin. (Engl. Ser.)*, **23** (2007), 189–192.

[12] Y. Su, Quasifinite representations of a Lie algebra of Block type, *J. Algebra*, **276** (2004), 117–128.

[13] Y. Su and X. Xu, Structure of divergence-free Lie algebras, *J. Algebra*, **243** (2001), 557–595.

[14] Y. Su and J. Zhou, Structure of the Lie algebras related to those of block, *Comm. Algebra*, **30** (2002), 3205–3226.

[15] X. Xu, New generalized simple Lie algebras of Cartan type over a field with characteristic 0, *J. Algebra*, **224** (2000), 23–58.

[16] X. Xu, Generalizations of Block algebras, *Manuscripta Math.*, **100** (1999), 489–518.

[17] Y. Wu, G. Song and Y. Su, Lie bialgebras of generalized Virasoro-like type, *Acta Math. Sin. (Engl. Ser.)*, **22** (2006), 1915–1922.

[18] X. Yue and Y. Su, Highest weight representations of a family of Lie algebras of Block type, *Acta Math. Sin. (Engl. Ser.)*, **24** (2008), 687–696.

Received April 14, 2009
Revised October 03, 2009