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The Pythagorean Theorem

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Pythagorean Theorem

Ask anyone you happen to meet socially what theorem he/she remembers from high-school geometry. The chances are high that the reply would be the Pythagorean theorem. But if they are asked to state the theorem they may not be able to do so except, perhaps, to say that it concerns triangles and that the conclusion is $c^2 = a^2+b^2$.

It is believed that Pythagoras lived in the 6^{th} century B. C. in the island of Samos, Egypt and in Croton in southern Italy, besides visits to neighbouring countries. He was the leader of a society, called Brotherhood, which was devoted to the study of mathematics, astronomy, religion and music. Among other things he is famous for the theorem attributed to him. The theorem was known earlier in some form or other in India and China; the Hindu mathematician Baudhayana discussed it around 800 B.C. in his book *Baudhayana Sulba Sutra*. It was known even earlier to the Babylonians.

Pythagoras seems to have been the one who formulated it in a form such that he is considered as the first pure mathematician in history. The theorem is called Pythagorean sometimes due to the secretive nature of his society. There is an interesting story of how he happened to notice the truth of the theorem. While waiting in a palace to be received by the king, his attention was drawn to the stone-square tiling of the floor. He imagined right angled triangles in half-squares implicit in the tiling together with the squares on its sides. It occurred to him that the area of the square over the hypotenuse of the right triangle is equal to the sum of the areas of the squares over the other two sides. Legend has it that Pythagorean brotherhood celebrated this discovery by sacrificing to gods a hetacomb (100 heads) of oxen! Regarding this, C. L. Dodgson (Lewis Carroll) wrote: "One can imagine oneself, even in these degenerate days, marking the epoch of some brilliant discovery by inviting a convivial friend or two, to join one in a beefsteak and a bottle of wine. But a hecatomb of oxen! It would produce a quite inconvenient supply of beef."

Is there a theorem in Euclidean geometry that has the most number of proofs?

Yes: the Pythagorean theorem. There seems to be about 500 different proofs of this theorem. The usual texts in geometry for high schools give the theorem in the form stated by Euclid, Book I, (Proposition 47): "In a right-angled triangle the square on the side subtending the right angle is equal to the squares on the sides containing the right angle" (Figure 1).

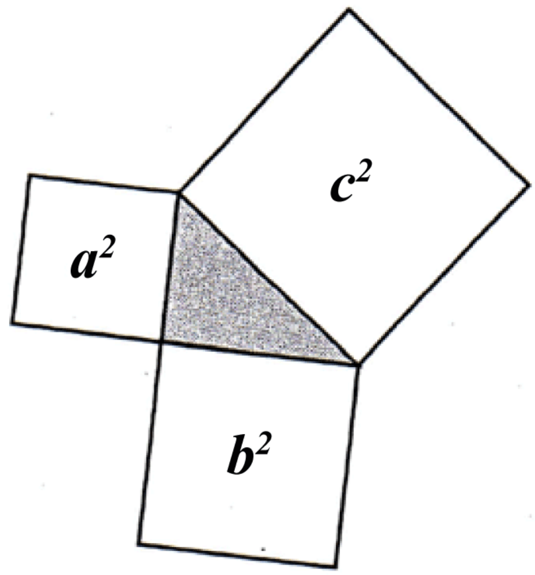

Figure 1: A right-angled triangle with integer sides is called a Pythagorean triangle (PT).

Such a triangle with sides 3, 4 and 5, with the hypotenuse of length 5, is the only PT with the three sides as consecutive numbers, and also the only triangle with the sum of the sides (=12) which is double its area (=6). The next PT with consecutive leg lengths has sides 21, 20, 29. The tenth such triangle has large sides: 27304196, 27304197, 38613965.

Anecdotes: It is very interesting to read how famous persons reacted to the theorem when they first heard about it:

When Albert Einstein was about 11 years old his uncle, Jacob Einstein, taught him the typical proof of the Pythagorean theorem. Young Einstein wondered if it was really necessary to have all those extra lines, angles, and squares in addition to the basic right triangle with hypotenuse c, and sides a and b. After "a little thinking" the sharp youngster came up with a proof that required only one additional line, the altitude above the hypotenuse. This height divides the large triangle into two smaller triangles that are similar to each other and also similar to the large triangle. This story is mentioned by Manfred Schroeder [1] who says in a footnote that "I have the story from Schneior Lifson of the Weizmann Institute in Tel Aviv, who has it from Einstein's assistant Ernst Strauss, to whom it was told by Albert himself."

Bertrand Russell writes in In *Praise of Idleness* (1935): Everyone knows the story of Hobbes's first contact with Euclid: opening the book, by chance, at the theorem of Pythagoras, he exclaimed, "By God, this is impossible," and proceeded to read the proofs backwards, until reaching the axioms, he became convinced. No one can doubt that this was for him a voluptuous moment, unsullied by the thought of the utility of geometry in measuring fields."

A group of representatives from the state of Utah in U. S. A. was in the congressional cafeteria during a break. Mr. James A. Garfield was one of them. He suggested, just to pass time, that they look at the Pythagorean theorem. Even though they knew that this theorem is a well-studied one, they tried to attempt a new proof. And they succeeded. Together with his colleagues Garfield worked for a little while and discovered that if the right triangle is cross-hatched with another copy of it such that they have one vertex (not at the right angle) common, so that a trapezium can be formed by joining the other vertices (not at the right angle), consisting of three triangles (Figure 2).

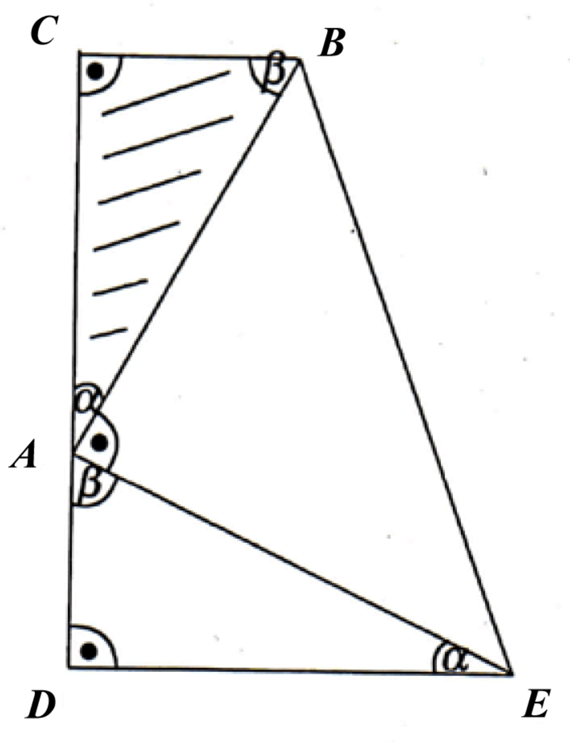

Figure 2: The trapezium has two right angles at two of its edges.

By calculating the area of this trapezium in two different ways, the conclusion of the Pythagorean theorem is obtained. Garfield's proof was published in the New England Journal of Education. Since then Garfield became the President of the U. S. A. Unfortunately, he was shot with a pistol by a crazy person in Washington's train station within a year of his presidency. This story is told in the Preface of the book by Norbert Hermann [2].

Raymond Smullyan [3] writes: "I had to present the Pythagorean theorem to a class in geometry. I drew a right triangle on the board with squares on the hypotenuse and legs and said, 'Obviously, the square on the hypotenuse has a larger area than either of the other two squares. Now suppose these three squares are made of beaten gold, and you were offered either the one large square or the two small squares. Which would you choose?' Interestingly enough, about half the class opted for the one large square, and the other half for the two smaller ones. A lively argument began. Both groups were equally amazed when told that it would make no difference."

The converse of the theorem (Euclid I, Proposition 48) is also true: If the square of the length of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right-angled triangle.

For proving this, construct a right-angled triangle A′B′C′ with legs a and b, by first constructing the line segment B′C′ and then at C′, constructing a perpendicular line segment congruent to AC. Let the hypotenuse of the triangle A′B′C′ have length d. Then by the Pythagorean theorem,

$a^2 + b^2 = d^2$. Since both c and d are positive, c = d. Consequently the two triangles are congruent. Now, since the angle B′C′A′ is a right angle, the angle BCA must also be a right angle.

There is a statue of Pythagoras in the Cathedral of Notre Dame in Chartes, France. A noteworthy fact about Pythagoras is that he was left-handed.

What is the importance of the Pythagorean Theorem?

It is important because it reveals a fundamental truth about the nature of space. It implies that space is flat, as opposed to being curved. On the surface of a globe or a bagel, the theorem needs modification. Einstein confronted this challenge in his general theory of relativity, where gravity is no longer viewed as a force, but rather as a manifestation of the curvature of space.

However, there is a version of the Pythagorean Theorem for right triangles on spheres.

A great circle on a sphere is any circle whose center coincides with the center of the sphere. A spherical triangle is any 3-sided region enclosed by sides that are arcs of great circles. If one of the corner angles is a right angle, the triangle is a spherical right triangle.

In such a triangle, let C denote the length of the side opposite right angle. Let A and B denote the lengths of the other two sides. Let R denote the radius of the sphere. Then the following equation holds:

$\cos(C/R) = \cos(A/R)\cos(B/R)$.

This equation is called the 'Spherical Pythagorean Theorem'. The usual Pythagorean Theorem can be obtained as a special case by letting R go to infinity, and expanding the cosines using Taylor series, and simplifying we obtain $C^2=A^2+B^2$. Indeed, as R goes to infinity, spherical geometry becomes Euclidean geometry. [http://www.math.hmc.edu/funfacts] [4].

Applications: There are many applications of the Pythagorean theorem. Alfred Posamentier [5] devotes a whole chapter presenting many applications. He begins with the following simple example: suppose one needs to buy a circular tabletop and wanted to know if it will fit through the doorway of the house. To find this out the diagonal length of the doorway is computed using the Pythagorean theorem and one simply checks how this compares with the diameter. Other applications discussed include: constructing irrational number lengths; determining if an angle is obtuse or acute; amazing geometric

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relationships; Pythagorean magic squares; extending the theorem to three dimensions and its applications.

Generalizations of the Pythagoras Theorem: Mathematics grows by extending known theorems to more general situations.

The 'square' in the statement of the theorem can be replaced by any similar figure, drawn appropriately, on all the sides of the right triangle; for example, consider a semi-circle on each of the sides; in the following Figure 3,

Area P = area Q + area R.

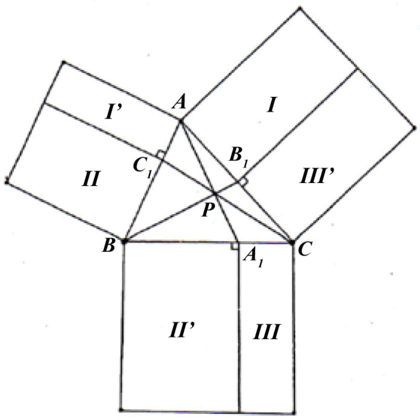

Figure 3: Euclid himself gave a generalization of this theorem.

Euclid Book VI. Proposition VI.31 says "In a right angled triangle the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Apollonius of Perga (circa 2nd century B.C.) is credited with extending the theorem to any triangle, not necessarily right-angled, with the result that

$c^2 = a^2 + b^2 - 2ab \cos C.$

This result is called the law of cosines. The Pythagorean theorem is usually derived by using this law. But the Pythagorean theorem itself is a special case of this law: if $C = \pi/2$, we have $c^2 = a^2 + b^2$. Thus the law of cosines is equivalent to the Pythagorean theorem.

An interesting generalization of the Pythagorean theorem is given by G. D. Chakerian and M. S. Klamkin [6] as follows:

Theorem 1: If P is any point in the interior of a triangle ABC, we consider the squares on the sides and the rectangles formed by extending the lines joining P to the vertices A, B and C in the respective squares (Figure 4).

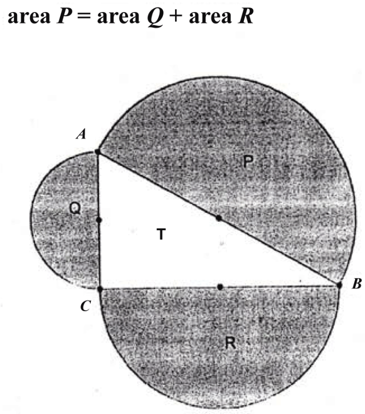

Figure 4: Label the rectangles I, I′, II, II′, III, III′ as shown in the figure.

Then the conclusion is: I = I′, II = II′, III = III′ if and only if P is the orthocenter of the triangle ABC.

The rectangles with the same labels in the figure have the same areas.

The Pythagorean theorem is the limiting case where the angle A is a right angle, in which case, the rectangles formed by the extended altitudes degenerate into line segments.

The following is from John dePillis [7], who attributes them to Robert Osserman of MSRI, Berkeley.

Question: Which gives more pizza: the small pizza + the medium pizza, or the large pizza?

Answer: Cut each pizza in half and Form the triangle whose sides are the respective diameters a, b, and c shown in the figure. If the angle C is a right angle, then $c^2=a^2 + b^2$, which means that the area of the small pizza + the area of the medium one = the area of the large pizza. The large pizza has a larger area than the other two pizzas once the hypotenuse increases, i.e., once the angle C>90°. Similarly it has a smaller area than the other two if C < 90° (Figure 5).

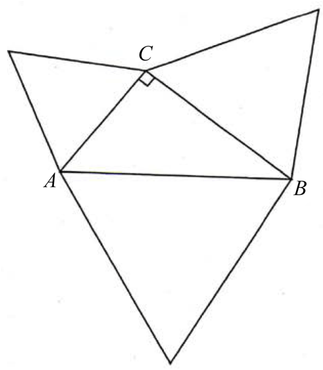

Figure 5: Pictorial representation of the Pythagorean Theorem.

Bride's chair: The pictorial representation of the Pythagorean Theorem is known in mathematical folklore under many names, the *bride's chair*, being probably the most popular, but also as the *Franciscan's cowl*, the *peacock's tail* and the *windmill*. Details are given in the website [4].

Theorem 2: If the vertices of the squares in the figure for the Pythagorean Theorem are joined to form three triangles, then these triangles have the same area.

Proof: The area of the triangle at vertex C = ½ab sin (180 – α) = area of triangle ABC. Similarly the areas of the triangles at the other two vertices are each equal to the area of the triangle ABC.

Finally, attention is drawn to an interesting chapter with the title: Many cheerful facts about the square of the hypotenuse, in a book by J. L. Heilbron [8-13]. Among other things, it is pointed out in this book that Euclid's proof (I.47) is called 'windmill' by several groups of students because of the resemblance of the classical figure (in Euclid with several lines) to the sails of a mill; its ingenious purely geometrical demonstration uses literal squares drawn on the sides of the right angle.

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