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Short Commentary

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The Spaces of Entire Function of Finite Order

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Abstract

This paper is a continuation of the research of the first author. We consider the linear topology space of entire functions of a proximate order and normal type with respect to the proximate order. We obtain the form of continuous linear functional on this space.

Keywords: Entire function; Proximate order; Normal type; Continuous linear functional

Introduction

This paper is a continuation of the research [1] where the linear topology space of entire functions of a proximate order and normal type, less than or equal σ, with respect to the proximate order were considered. We introduce the necessary definitions. A function $\rho(r)$, defined on the ray $(0,\infty)$ and satisfying the Lipschitz condition on any segment $[a, b]\subset(0,\infty)$ that satisfies the conditions

$$\lim_{r\to\infty}\rho(r)=\rho\geq 0,\ \text{and}\ \lim_{r\to\infty} r\rho'(r)\ln r=0$$

This is called *a proximate order*.

A detailed exposition of the properties of proximate order can be found [2,3]. In this paper we use the notation $V(r)=r^{\rho(r)}$. We will assume that $V(r)$ is an increasing function on $(0,\infty)$ and $\lim_{r\to+0}V(r)=0$.

We now formulate some simple property of proximate order that we shall need frequently [2].

For $r\to\infty$ *and* $0<a\leq k\leq b<\infty$ *asymptotic inequality holds uniformly in k.*

$$(1-\varepsilon)k^{\rho}V(r)<V(kr)<(1+\varepsilon)k^{\rho}V(r) \qquad (1)$$

Let $M_f(r)=\max_{|z|=r}|f(z)|$. If for the entire function $f(z)$ the quantity

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M_f(r)}{V(r)}$$

Is different from zero and infinity, then $\rho(r)$ is called of *a proximate order of the given entire function f(z) and σf is called the type of the function f(z) with respect to the proximate order* $\rho(r)$.Let $\rho(r)$ be a proximate order, $\lim_{x\to\infty}\rho(r)=\rho\geq 0$. A single valued function $f(z)$ of the complex variable z is said to belong to the space $[\rho(r),¥)$ if $f(z)$ has the order less than $\rho(r)$ or equal $\rho(r)$ but in this case type less than ¥. A sequence of functions $\{f_n(z)\}$ from $[\rho(r),¥)$ converges in the sense of $[\rho(r),¥)$ if

(i) It converges uniformly on compacts, (ii) there exists $\beta<1$ such that

$$|f_n(z)|<C(\beta)\exp[\beta V|z|)],\ |z|>r_0(\beta)(n\geq 1),$$

where $r_0(\beta)$ does not depend on $(n\geq 1)$. For a suitable $C(\beta)$, which does not depend on n, for all z

$$|f_n(z)|<C(\beta)\exp[\beta V|z|)]\quad(n\geq 1) \qquad (2)$$

The space $[\rho(r),¥)$ is the linear topology space with sequence topology. Furthermore, a single valued function $f(z)$ of the complex variable z is said to belong to the space $[\rho(r), p]$ if $f(z)$ has the order less than $\rho(r)$ or equal $\rho(r)$ but in this case type less than or equal p. A sequence of functions $\{f_n(z)\}$ from $[\rho(r), p]$ converges in the sense of $[\rho(r), p]$ if (i) it converges uniformly on compacts, (ii) for all $\varepsilon>0$ there exists $r_0(\varepsilon)$ does not depend on n such that

$$|f_n(z)|<\exp[(p+\varepsilon)V|z|)],\ |z|>r_0(\varepsilon)(n\geq 1).$$

The space $[\rho(r), p]$ is also the linear topology space with sequence topology. We introduce the function $\varphi(t)$ defined to be the unique solution of the equation $t=V(r)$. So

$\varphi(V(t))=t.$ (3)

Theorem 1.1 ([2, Theorem 2', p.42])

The type σf of the entire function $f(z)=\sum_{n=0}^{\infty}c_nz^n$ *with the proximate order* $\rho(r)$ $(\rho>0)$ *is given by the equation*

$$\limsup_{n\to\infty}\varphi(n)^n\sqrt[n]{|c_n|}=(e\sigma f\rho)^{1\backslash\rho} \qquad (4)$$

Let $\rho>0$

$$d_n=\frac{(e\sigma\rho)^{n\backslash\rho}}{(\varphi(n))^n}\ n\geq 1,\ d_0=1$$

For a function $f(z)=\sum_{n=0}^{\infty}c_nz^n\in[\rho(r),p]$ we associate the function

$$f(z)=\sum_{n=0}^{\infty}b_nz^n,\ b_n=\frac{c_n}{d_n}(n\geq 0) \qquad (5)$$

It is regular, in any case in the circle $|z|<1$ [1]. Fact mapping function $f(z)$ of $[\rho(r), p]$ to the function $F(z)$ as indicated above will be celebrating a record $f(z)\sim F(z)$.

In [1] it is proved the following two theorems.

Theorem 1.2

In order to be a sequence $\{f_n(z)\}$ *of functions from* $[\rho(r), p]$ *to converge in the sense of* $[\rho(r), p]$ *necessary and sufficient that the sequence* $\{f_n(z)\}$ *(fn(z)~Fn(z)) converges uniformly inside the disk* $|z|<1$.

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Theorem 1.3

Continuous linear functional l on the space [$\rho(r)$, p] *has the form*

$$l(f)=\sum_{n=0}^{\infty}a_nc_n,\ f(z)=\sum_{n=0}^{\infty}c_nz^n, \tag{6}$$

Where the quantities an satisfy

$$\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}=0 \tag{7}$$

The following is our main result.

Theorem 1.4

Continuous linear functional l on the space [$\rho(r)$,¥ *has the form*

$$l(f)=\sum_{n=0}^{\infty}a_nc_n,\ f(z)=\sum_{n=0}^{\infty}c_nz^n \tag{8}$$

Where the quantities an satisfy

$$\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}=0 \tag{9}$$

The Space of Entire Functions [$\rho(r)$,¥)

We now prove the theorem 1.4. Let $l(f)$ be a continuous linear functional on the space [$\rho(r)$,¥). Set $l(z^n)=a_n(n\geq 0)$ Let $f(z)=\sum_{n=0}^{\infty}c_nz^n$ be a function in [$\rho(r)$,¥). Since the series converges in the sense of [$\rho(r)$,¥) then, by continuous of the functional,

$$l(f)=\sum_{n=0}^{\infty}c_nl(z^n)=\sum_{n=0}^{\infty}c_na_n$$

Hence

$$l(f)=\sum_{n=0}^{\infty}a_nc_n,\ f(z)=\sum_{n=0}^{\infty}c_nz^n \tag{10}$$

Take an arbitrary finite p>0. Functional $l(f)$ is, in particular, continuous linear functional on the space [$\rho(r)$, p]. By theorem 1.3, the condition

$$\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}<(epp)^{-1/\rho}$$

But p is arbitrary, hence,

$$\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}=0$$

We now verify that if the condition (9) then the functional (10) is continuous linear functional on the space [$\rho(r)$,¥). Let $f(z)=\sum_{n=0}^{\infty}c_nz^n\in[\rho(r),\infty]$ By theorem 1.1, $\limsup_{n\to\infty}\varphi(n)\sqrt[n]{|c_n|}=(e\sigma f\rho)^{-1/\rho}<\infty$. Then $\limsup_{n\to\infty}\sqrt[n]{|a_nc_n|}=\limsup_{n\to\infty}\varphi(n)\sqrt[n]{|c_n|}\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}=0$

And then the series (10) converges.

Let $\{f_k(z)=\sum_{n=0}^{\infty}c_n^{(k)}z^n\subset[\rho(r),\infty)$ $f_k(z)\to f(z)=\sum_{n=0}^{\infty}c_nz^n$ if k→¥ and let l satisfies (9). By (2), there exists $\beta>0$ such that {{$f_k(z)$}, $f(z)$}Ì[$r(r)$, β] in the sense [$\rho(r)$, β]. By (9) and (8) l is continuous linear functional on the space [$\rho(r)$, β]. Then $l(f_k)$→l $l(f)$ if k→¥. Therefore l is continuous linear functional on the space [$\rho(r)$,¥).

Space of Entire Functions $E_\rho(r)$

We now consider the space of entire functions $E_{\rho(r)}$ which have a proximate orders less then $\rho(r)$. A proximate order $\rho_1(r)$ less then $\rho(r)$ if $0<\lim_{r\to\infty}\rho(r)=\rho_1<\lim_{r\to\infty}\rho(r)=\rho A$ single valued function $f(z)$ of the complex variable z is said to belong to the space $E_{\rho(r)}$ if $f(z)$ has the order less than $\rho(r)$.

A sequence of functions $\{f_n(z)\}$ from $E_{\rho(r)}$ converges in the sense of $E_{\rho(r)}$ if (i) it converges uniformly on compacts, (ii) there exists proximate order $\rho_1(r)$, $0<\lim_{r\to\infty}\rho_1(r)=\rho_1<\lim_{r\to\infty}\rho(r)=\rho$ such that

$$|f_n(z)|<\exp[V_1|z|)],\ |z|>r_0(\beta)(n\geq 1) \tag{11}$$

where $r_0(\beta)$ does not depend on ($n\geq 1$), $V_1(r)=r^{r1(r)}$.The space $E_{\rho(r)}$ is the linear topology space with sequence topology. A continuous linear functional $l(f)$ on the space $E_{\rho(r)}$ has the form (8). Let us find the conditions that satisfy the values a_n. The functional $l(f)$ is in particular continuous linear functional on the space [$\rho1(r)$,¥) for all proximate order $\rho_1(r)$, $0<\lim_{r\to\infty}\rho_1(r)=\rho_1<\lim_{r\to\infty}\rho(r)=\rho$. Therefore, by theorem

$$\limsup_{n\to\infty}\varphi_1^{-1}(n)\sqrt[n]{|a_n|}=0, \tag{12}$$

where $\varphi_1(t)$ defined to be the unique solution of the equation $t=V_1(r)$. From this

$$\frac{\log|a_n|}{\varphi_1(n)n}<1,\ n>n_0$$

So $\rho_1(r)$ is arbitrary less then $\rho(r)$ that

$$\limsup_{n\to\infty}\frac{\log|a_n|}{\varphi(n)n}\leq 1 \tag{13}$$

Contrary, let the condition (13) is true and $\rho1(r)$ is arbitrary less than $\rho(r)$. So $\varphi_1(n)>\varphi(n)$, $n>n_0$, that

$$\frac{\log|a_n|}{\varphi_1(n)n}<1,\ n>n_0$$

Therefor the condition (12) is true and $l(f)$ is continuous linear functional on the space [$\rho(r)$,¥). So $\rho_1(r)$ is arbitrary less then, $\rho(r)$ that $l(f)$ is continuous linear functional on the space $E_{r(r)}$.

Theorem 3.1

Continuous linear functional l on the space $E_{\rho(r)}$ *has the form*

$$l(f)=\sum_{n=0}^{\infty}a_nc_n,\ f(z)=\sum_{n=0}^{\infty}c_nz^n$$

Where the quantities an satisfy

$$\limsup_{n\to\infty}\frac{\log|a_n|}{\varphi(n)n}\leq 1$$

Remark: The case of the spaces [ρ,¥) and E_r, where $\rho(r)=\rho>0$, considered A.F Leont'ev [4].

Conclusion

The linear topology space of entire functions of a proximate order and normal type with respect to the proximate order is considered. We obtain the form of continuous linear functional on this space through our work.

References

1. Kruglikov B, Lynchagin V, Straume E (2009) Differential Equations - Geometry, Symmetries and Integrability. Springer, New York.
2. Mikhailov AV (2009) Integrability: Lecture Notes in Physics. Springer, New York.
3. Blackmore D, Prykarpaysky AK, Samoy Lenko VH (2011) Nonlinear Dynamical

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Systems of Mathematical Physics: Spectral and Simplectic Integrability Analysis. World Scientific Publishing.

4. Wazwaz AM (2009) Partial Differential Equations and Solitary Waves Theory. Springer, New York.